Abstract—The $n$-dimensional hypercube network $Q_n$ is one of the most popular interconnection networks since it has simple structure and is easy to implement. The $n$-dimensional augmented cube, denoted by $AQ_n$, an important variation of the hypercube, possesses several embedding properties that hypercubes and other variations do not possess. The advantages of $AQ_n$ are that the diameter is only about half of the diameter of $Q_n$ and they are node-symmetric. Recently, some interesting properties of $AQ_n$ were investigated. A graph $G$ contains two-equal path partition if for any two distinct pairs of nodes $(u, s)$ and $(v, r)$ of $G$, there exist two node-disjoint paths $P$ and $Q$ joining $u$ and $s$, and $v$ and $r$, respectively. We then show that the $n$-dimensional augmented cube $AQ_n$, with $n \geq 2$, contains two-equal path partition.

Index Terms—edge-disjoint Hamiltonian cycles, two-equal path partition, augmented cubes, hypercubes, parallel computing

I. INTRODUCTION

PARALLEL computing is important for speeding up computation. The design of an interconnection network is the first thing to be considered. Many topologies have been proposed in the literature [3], [7], [8], [9], [10], and the desirable properties of an interconnection network include symmetry, relatively small degree, small diameter, embedding capabilities, scalability, robustness, and efficient routing. Among those proposed interconnection networks, the hypercube is a popular interconnection network with many attractive properties such as regularity, symmetry, small diameter, strong connectivity, recursive construction, partition ability, and relatively low link complexity [24]. The architecture of an interconnection network is usually modeled by a graph, where the nodes represent the processing elements and the edges represent the communication links. In this paper, we will use graphs and networks interchangeably.

The $n$-dimensional augmented cube $AQ_n$ was first proposed by Choudum et al. [6] and possesses some properties superior to the hypercube. The diameter of augmented cubes is only about half of the diameter of hypercubes and augmented cubes are node-symmetric [6]. Recently, some interesting properties, such as conditional link faults, of the augmented cube $AQ_n$ were investigated. Choudum and Sunitha proved $AQ_n$, with $n \geq 2$, is pancyclic, that is, $AQ_n$ contains cycles of arbitrary length [6]. Hsu et al. considered the fault hamiltonicity and the fault hamiltonian connectivity of the augmented cube $AQ_n$ [13]. Wang et al. showed that $AQ_n$, with $n \geq 4$, remains pancyclic provided faulty vertices and/or edges do not exceed $2n - 3$ [26]. Hsieh and Shiu proved that $AQ_n$ is node-pancyclic, in which for every node $u$ and any integer $l \geq 3$, the graph contains a cycle $C$ of length $l$ such that $u$ is in $C$ [11]. Hsu et al. proved that $AQ_n$ is geodesic pancyclic and balanced pancyclic [14]. Recently, Chan et al. [5] improved the results in [14] to obtain a stronger result for geodesic-pancyclic and fault-tolerant panconnectivity of the augmented cube $AQ_n$. In [19], Ma et al. proved that $AQ_n$ contains paths between any two distinct vertices of all lengths from their distance to $2^n - 1$; and that $AQ_n$ still contains cycles of all lengths from 3 to $2^n$ when any $(2n - 3)$ edges are removed from $AQ_n$. Xu et al. determined the vertex and the edge forwarding indices of $AQ_n$ as $2^n/9 + (1)^{n+1}/9 + 2^n/3 - 2^n + 1$ and $2^n - 1$, respectively [27]. Chan computed the distinguishing number of the augmented cube $AQ_n$ [4]. Lee et al. studied the Hamiltonian path problem on $AQ_n$ with a required node being the end node of a Hamiltonian path [18].

Two Hamiltonian cycles in a graph are said to be edge-disjoint if they do not share any common edge. The edge-disjoint Hamiltonian cycles can provide advantage for algorithms that make use of a ring structure [25]. The following application about edge-disjoint Hamiltonian cycles can be found in [25]. Consider the problem of all-to-all broadcasting in which each node sends an identical message to all other nodes in the network. There is a simple solution for the problem using an $N$-node ring that requires $N - 1$ steps, i.e., at each step, every node receives a new message from its ring predecessor and passes the previous message to its ring successor. If the network admits edge-disjoint rings, then messages can be divided and the parts broadcast along different rings without any edge contention. If the network can be decomposed into edge-disjoint Hamiltonian cycles, then the message traffic will be evenly distributed across all communication links. Edge-disjoint Hamiltonian cycles also form the basis of an efficient all-to-all broadcasting algorithm for networks that employ warmhole or cut-through routing [17].

The edge-disjoint Hamiltonian cycles in $k$-ary $n$-cubes has been constructed in [1]. Barden et al. constructed the maximum number of edge-disjoint spanning trees in a hypercube [2]. Petrovic et al. characterized the number of edge-disjoint Hamiltonian cycles in hyper-tournaments [23]. Hsieh et al. constructed edge-disjoint spanning trees in locally twisted cubes [12]. The existence of a Hamiltonian cycle in augmented cubes has been shown [6], [13]. However, there has been little work reported so far on edge-disjoint properties in the augmented cubes. In this paper, we use a recursive construction to show that, for any integer $n \geq 3$, there are two edge-disjoint Hamiltonian cycles in the $n$-

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dimensional augmented cube \(AQ_n\).

Finding node-disjoint paths is one of the important issues of routing among nodes in various interconnection networks. Node-disjoint paths can be used to avoid communication congestion and provide parallel paths for an efficient data routing among nodes. Moreover, multiple node-disjoint paths can be more fault-tolerant of nodes or link failures and greatly enhance the transmission reliability. A path partition of a graph \(G\) is a family of node-disjoint paths that contains all nodes of \(G\). For an embedding of linear arrays in a network, the partition implies every node can be participated in a pipeline computation. Finding a path partition and its variants of a graph has been investigated [15], [16], [20], [21], [22]. In this paper, we study a variation of path partition, called two-equal path partition. A graph \(G\) contains two-equal path partition if for any two distinct pairs of nodes \((u_s, u_t)\) and \((v_s, v_t)\) of \(G\), there exists a path partition \(\{P, Q\}\) of \(G\) such that (1) \(P\) joins \(u_s\) and \(u_t\), (2) \(Q\) joins \(v_s\) and \(v_t\), and (3) \(|P| = |Q|\). In this paper, we will show that the augmented cube \(AQ_n\), with \(n \geq 2\), contains two-equal path partition.

The rest of the paper is organized as follows. In Section II, the structure of the augmented cube is introduced, and some definitions and notations used in this paper are given. Section III shows the construction of two edge-disjoint Hamiltonian cycles in the augmented cubes. In Section IV, we show that augmented cubes contain two-equal path partition. Finally, we conclude this paper in Section V.

II. Preliminaries

We usually use a graph to represent the topology of an interconnection network. A graph \(G = (V, E)\) is a pair of the node set \(V\) and the edge set \(E\), where \(V\) is a finite set and \(E\) is a subset of \(\{(u, v)|(u, v)\}\) of \(V\). We will use \(V(G)\) and \(E(G)\) to denote the node set and the edge set of \(G\), respectively. If \((u, v)\) is an edge in a graph \(G\), we say that \(u\) is adjacent to \(v\). A neighbor of a node \(v\) in a graph \(G\) is any node that is adjacent to \(v\). Moreover, we use \(NC_G(v)\) to denote the set of neighbors of \(v\) in \(G\). The subscript \(G\) of \(NC_G(v)\) can be removed from the notation if it has no ambiguity.

A path \(P\), represented by \(\langle v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_{t-1}\rangle\), is a sequence of distinct nodes such that two consecutive nodes are adjacent. The first node \(v_0\) and the last node \(v_{t-1}\) visited by \(P\) are called the path-start and path-end of \(P\), denoted by \(start(P)\) and \(end(P)\), respectively, and they are called the end nodes of \(P\). Path \(\langle v_{t-1} \rightarrow \cdots \rightarrow v_1 \rightarrow v_0 \rangle\) is called the reversed path, denoted by \(rev(P)\), of \(P\). That is, \(rev(P)\) visits the nodes of \(P\) from \(end(P)\) to \(start(P)\) sequentially. In addition, \(P\) is a cycle if \(|V(P)| \geq 3\) and \(end(P)\) is adjacent to \(start(P)\). A path (or cycle) in \(G\) is called a Hamiltonian path (or Hamiltonian cycle) if it contains every node of \(G\) exactly once. A graph \(G\) is Hamiltonian connected if, for any two distinct nodes \(u, v\), there exists a Hamiltonian path with end nodes \(u, v\). Two paths (or cycles) \(P_1\) and \(P_2\) connecting a node \(u\) to a node \(v\) are said to be edge-disjoint if \(E(P_1) \cap E(P_2) = \emptyset\). Two paths (or cycles) \(Q_1\) and \(Q_2\) of graph \(G\) are called node-disjoint iff \(V(Q_1) \cap V(Q_2) = \emptyset\). Two node-disjoint paths \(Q_1\) and \(Q_2\) can be concatenated into a path, denoted by \(Q_1 \Rightarrow Q_2\), if \(end(Q_1)\) is adjacent to \(start(Q_2)\).

Definition 1. A graph \(G\) contains two-equal path partition if for any two distinct pairs of nodes \((u_s, u_t)\) and \((v_s, v_t)\) of \(G\), there exist two node-disjoint paths \(P\) and \(Q\) satisfying that (1) \(start(P) = u_s\) and \(end(P) = u_t\), (2) \(start(Q) = v_s\) and \(end(Q) = v_t\), (3) \(|P| = |Q|\), and (4) \(V(P) \cup V(Q) = V(G)\).

Now, we introduce augmented cubes. The node set of the \(n\)-dimensional augmented cube \(AQ_n\) is the set of binary strings of length \(n\). A binary string \(b\) of length \(n\) is denoted by \(b_{n-1}b_{n-2}\cdots b_0\), where \(b_{n-1}\) is the most significant bit. We denote the complement of bit \(b_i\) by \(\overline{b_i} = 1 - b_i\) and the leftmost bit complement of binary string \(b\) by \(\overline{b} = \overline{b_{n-1}}\overline{b_{n-2}}\cdots \overline{b_0}\). We then give the recursive definition of the \(n\)-dimensional augmented cube \(AQ_n\), with integer \(n \geq 1\), as follows.

Definition 2. [6] Let \(n \geq 1\). The \(n\)-dimensional augmented cube, denoted by \(AQ_n\), is defined recursively as follows.

(1) \(AQ_1\) is a complete graph \(K_2\) with the node set \([0, 1]\).

(2) For \(n \geq 2\), \(AQ_n\) is built from two disjoint copies \(AQ_{n-1}\) according to the following steps. Let \(AQ_{n-1}\) denote the graph obtained by prefixing the label of each node of one copy of \(AQ_{n-1}\) with 0, let \(AQ_{n-1}^{1}\) denote the graph obtained by prefixing the label of each node of the other copy of \(AQ_{n-1}\) with 1. Then, adding \(2^n\) edges between \(AQ_{n-1}\) and \(AQ_{n-1}^{1}\) by the following rule. A node \(b = b_{n-2}b_{n-3}\cdots b_0\) of \(AQ_{n-1}\) is adjacent to a node \(a = a_{n-2}a_{n-3}\cdots a_0\) of \(AQ_{n-1}^{1}\) iff either

(i) \(a_i = b_i\) for all \(n - 2 \geq i \geq 0\) (in this case, \((b, a)\) is called a hypercube edge), or

(ii)\(a_i = \overline{b_i}\) for all \(n - 2 \geq i \geq 0\) (in this case, \((b, a)\) is called a complement edge).

It was proved in [6] that \(AQ_n\) is node transitive, \((2n - 1)\)-regular, and has diameter \([\frac{n}{2}]\). According to Definition 2, \(AQ_n\) contains \(2^n\) nodes. Further, \(AQ_{n}\) is decomposed into two sub-augmented cubes \(AQ_{n-1}^{0}\) and \(AQ_{n-1}^{1}\), where \(AQ_{n-1}\) consists of those nodes \(b\) with \(b_{n-1} = i\). For each \(i \in \{0, 1\}\), \(AQ_{n-1}^{i}\) is isomorphic to \(AQ_{n-1}\). For example, Fig. 1(a) shows \(AQ_2\) and Fig. 1(b) depicts \(AQ_3\) consisting of two sub-augmented cubes \(AQ_2^0, AQ_2^1\). The following proposition can be easily verified from Definition 2.

Proposition 1. Let \(AQ_n\) be the augmented cube decomposed into \(AQ_{n-1}^{0}\) and \(AQ_{n-1}^{1}\). For any \(b \in V(AQ_{n-1}^{i})\) and \(i \in \{0, 1\}, \overline{b} \in V(AQ_{n-1}^{1-i})\) and \(b \in V(b)\).

Let \(b\) be a binary string \(b_{n-1}b_{n-2}\cdots b_0\) of length \(n\). We
denote $b^i$ the new binary string obtained by repeating $b$ string $i$ times. For instance, $(10)^2 = 1010$ and $0^3 = 000$.

The following Hamiltonian connected property of the augmented cube can be proved by induction.

**Lemma 2.** For any integer $n \geq 2$, $AQ_n$ is Hamiltonian connected.

**Proof:** We prove this lemma by induction on $n$, the dimension of the augmented cube $AQ_n$. Obviously, $AQ_2$ is Hamiltonian connected since it is a complete graph with 4 nodes. Assume that $AQ_k$, with $k \geq 2$, is Hamiltonian connected. We will prove that $AQ_{k+1}$ is Hamiltonian connected. We first decompose $AQ_{k+1}$ into two sub-augmented cubes $AQ_k^{\text{up}}$ and $AQ_k^{\text{down}}$. Let $u, v$ be any two distinct nodes of $AQ_{k+1}$. There are two cases:

**Case 1:** $u, v \in V(AQ_k^{\text{up}})$, for $i \in \{0, 1\}$. By inductive hypothesis, there is a Hamiltonian path $P$ in $AQ_k^{\text{up}}$ with end nodes $u, v$. Let $P = u \rightarrow P'$ and let $\text{start}(P') = w$. By inductive hypothesis, there is a Hamiltonian path $Q$ in $AQ_k^{\text{down}}$ such that $\text{start}(Q) = \overline{w}$ and $\text{end}(Q) = \overline{w}$. By Proposition 1, $\overline{w} \in N(u)$ and $\overline{w} \in N(w)$. Then, $u \Rightarrow Q \Rightarrow P'$ is a Hamiltonian path of $AQ_{k+1}$ with end nodes $u, v$.

**Case 2:** $u, v \in V(AQ_k^{\text{down}})$, for $i \in \{0, 1\}$. Let $w$ be a node in $AQ_k^{\text{up}}$ such that $w \neq u$ and $\overline{w} \neq v$. By inductive hypothesis, there is a Hamiltonian path $P$ in $AQ_k^{\text{up}}$ such that $\text{start}(P) = u$ and $\text{end}(P) = w$. In addition, there is a Hamiltonian path $Q$ in $AQ_k^{\text{down}}$ such that $\text{start}(Q) = \pi$ and $\text{end}(Q) = \pi$. By Proposition 1, $\pi \in N(w)$. Then, $P \Rightarrow Q$ is a Hamiltonian path of $AQ_{k+1}$ with end nodes $u, v$.

By the above cases, $AQ_{k+1}$ is Hamiltonian connected. By induction, $AQ_n$, with $n \geq 2$, is Hamiltonian connected.

**III. Two Edge-disjoint Hamiltonian Cycles**

Obviously, $AQ_2$ has no two edge-disjoint Hamiltonian cycles since each node is incident to three edges. For any integer $n \geq 3$, we will construct two edge-disjoint Hamiltonian paths, $P$ and $Q$, in $AQ_n$ such that $\text{start}(P) = 000$, $\text{end}(P) = 100$, $\text{start}(Q) = 010$, and $\text{end}(Q) = 110$.

**Lemma 3.** There are two edge-disjoint Hamiltonian paths $P$ and $Q$ in $AQ_3$ such that $\text{start}(P) = 000$, $\text{end}(P) = 100$, $\text{start}(Q) = 010$, and $\text{end}(Q) = 110$.

**Proof:** We prove this lemma by constructing such two paths. Let $P = (000 \rightarrow 010 \rightarrow 011 \rightarrow 001 \rightarrow 101 \rightarrow 111 \rightarrow 110 \rightarrow 100)$, and let $Q = (010 \rightarrow 001 \rightarrow 000 \rightarrow 011 \rightarrow 110 \rightarrow 101 \rightarrow 110)$.

Fig. 2 depicts the constructions of $P$ and $Q$. Clearly, $P$ and $Q$ are edge-disjoint Hamiltonian paths in $AQ_3$.

By Proposition 1, nodes 000 and 100 are adjacent, and nodes 010 and 110 are adjacent. Thus, we have the following corollary.

**Corollary 4.** There are two edge-disjoint Hamiltonian cycles in $AQ_3$.

Using Lemma 3, we prove the following lemma.

**Lemma 5.** There are two edge-disjoint Hamiltonian paths $P$ and $Q$ in $AQ_4$ such that $\text{start}(P) = 0000$, $\text{end}(P) = 1000$, $\text{start}(Q) = 0010$, and $\text{end}(Q) = 1010$.

**Proof:** We first decompose $AQ_4$ into two sub-augmented cubes $AQ_3^{\text{up}}$ and $AQ_3^{\text{down}}$. By Lemma 3, there are two edge-disjoint Hamiltonian paths $P'$ and $Q'$, for $i \in \{0, 1\}$, in $AQ_3^i$ such that $\text{start}(P') = 0000$, $\text{end}(P') = 1000$, $\text{start}(Q') = 0010$, and $\text{end}(Q') = 1110$. By Proposition 1, we have that $\text{end}(P') \in N(\text{end}(P^{i}))$ and $\text{end}(Q') \in N(\text{end}(Q^{i}))$. Let $P = P^{0} \Rightarrow P^{rev}$ and let $Q = Q^{0} \Rightarrow Q^{rev}$, where $P^{rev}$ and $Q^{rev}$ are the reversed paths of $P$ and $Q$, respectively. Then, $P$ and $Q$ are two edge-disjoint Hamiltonian paths in $AQ_4$ such that $\text{start}(P) = 0000$, $\text{end}(P) = 1000$, $\text{start}(Q) = 0010$, and $\text{end}(Q) = 1010$. Fig. 3 shows the constructions of such two edge-disjoint Hamiltonian paths in $AQ_4$. Thus, the lemma holds true.

By Proposition 1, nodes 0000 and 1000 are adjacent, and nodes 0010 and 1010 are adjacent. The following corollary immediately holds true from Lemma 5.

**Corollary 6.** There are two edge-disjoint Hamiltonian cycles in $AQ_4$.

Based on Lemma 3, we prove the following lemma by the same arguments in proving Lemma 5.

**Lemma 7.** For any integer $n \geq 3$, there are two edge-disjoint Hamiltonian paths $P$ and $Q$ in $AQ_n$, such that $\text{start}(P) = 0000$, $\text{end}(P) = 1000$, $\text{start}(Q) = 0010$, and $\text{end}(Q) = 1010$.

**Proof:** We prove this lemma by induction on $n$, the dimension of the augmented cube. It follows from Lemma 3 that the lemma holds for $n = 3$. Suppose that the lemma is true for the case $n = k$ ($k \geq 3$). Assume that $n = k + 1$. We will prove the lemma holds when
n = k + 1. The proof is the same as that of Lemma 5. We first partition $AQ_{k+1}$ into two sub-augmented cubes $AQ_k^0$ and $AQ_k^1$. By the induction hypothesis, there are two edge-disjoint Hamiltonian paths $P^i$ and $Q^i$, for $i \in \{0, 1\}$, in $AQ_k^i$ such that $\text{start}(P^i) = i0(0)^{k-3}00$, $\text{end}(P^i) = i1(0)^{k-3}00$, $\text{start}(Q^i) = i0(0)^{k-3}10$, and $\text{end}(Q^i) = i1(0)^{k-3}10$. By Proposition 1, we get that

$$\text{end}(P^0) \in N(\text{end}(P^1)) \text{ and } \text{end}(Q^0) \in N(\text{end}(Q^1)).$$

Let $P = P^0 \Rightarrow P^1$ rev and let $Q = Q^0 \Rightarrow Q^1$ rev, where $P^1$ rev and $Q^1$ rev are the reversed paths of $P^1$ and $Q^1$, respectively. Then, $P$ and $Q$ are two edge-disjoint Hamiltonian paths in $AQ_{k+1}$ such that $\text{start}(P) = 0(0)^{k-3}00$, $\text{end}(P) = 1(0)^{k-3}00$, $\text{start}(Q) = 0(0)^{k-3}10$, and $\text{end}(Q) = 1(0)^{k-3}10$. Fig. 4 depicts the constructions of such two edge-disjoint Hamiltonian paths in $AQ_{k+1}$. Thus, the lemma holds true when $n = k + 1$. By induction, the lemma holds true.

By Proposition 1, nodes $\text{start}(P) = (0^n)^{n-3}00$ and $\text{end}(P) = 1(0)^{n-3}00$ are adjacent, and nodes $\text{start}(Q) = 0(0)^{n-3}10$ and $\text{end}(Q) = 1(0)^{n-3}10$ are adjacent. It immediately follows from Lemma 7 that the following theorem holds true.

**Theorem 8.** For any integer $n \geq 3$, there are two edge-disjoint Hamiltonian cycles in $AQ_n$.

IV. TWO-EQUAL PATH PARTITION

In this section, we will show that, for any $n \geq 2$, the $n$-dimensional augmented cube $AQ_n$ contains two-equal path partition. That is, for any two distinct pairs of nodes $(u_x, u_y)$ and $(v_x, v_y)$ of $AQ_n$, there exist two node-disjoint paths $P$ and $Q$ of $AQ_n$ satisfying that (1) $\text{start}(P) = u_x$ and $\text{end}(P) = u_y$, (2) $\text{start}(Q) = v_x$ and $\text{end}(Q) = v_y$, (3) $|P| = |Q|$, and (4) $V(P) \cup V(Q) = V(AQ_n)$. We will prove it by induction on $n$, the dimension of $AQ_n$. Initially, $AQ_2$ clearly contains two-equal path partition since it is a complete graph with four nodes.

**Lemma 9.** $AQ_2$ contains two-equal path partition.

Now, suppose that $AQ_k$, with $k \geq 2$, contains two-equal path partition. We will prove that $AQ_{k+1}$ contains two-equal path partition. First, we decompose $AQ_{k+1}$ into two sub-augmented cubes $AQ_k^0$ and $AQ_k^1$. Let $(u_x, u_y)$ and $(v_x, v_y)$ be any two pairs of distinct nodes in $AQ_k$. We will construct two node-disjoint paths $P$ and $Q$ of $AQ_{k+1}$ such that $P$ joins $u_x$ and $u_y$, $Q$ joins $v_x$ and $v_y$, and $|P| = |Q| = 2^k$. The are four cases:

*Case 1: $u_x, u_y, v_x, v_y$ are in the same sub-augmented cube.* Without loss of generality, assume that $u_x, u_y, v_x, v_y$ are in $AQ_k^0$. By inductive hypothesis, there is a path partition $\{P^0, Q^0\}$ of $AQ_k^0$ such that $|P^0| = |Q^0|$, $\text{start}(P^0) = u_x$, $\text{end}(P^0) = u_y$, $\text{start}(Q^0) = v_x$, and $\text{end}(Q^0) = v_y$. Let $P^0 = u_x \rightarrow P$ and $Q^0 = v_x \rightarrow Q$. Let $w_P = \text{start}(P^0)$ and let $w_Q = \text{start}(Q^0)$. Let $(\pi_x, \pi_P)$ and $(\pi_y, \pi_Q)$ be two pairs of distinct nodes in $AQ_k^0$. By inductive hypothesis, there are two node-disjoint paths $P^1$ and $Q^1$ of $AQ_k^1$ such that $|P^1| = |Q^1| = 2^{k-1}$, $\text{start}(P^1) = \pi_x$, $\text{end}(P^1) = \pi_P$, $\text{start}(Q^1) = \pi_y$, and $\text{end}(Q^1) = \pi_Q$. By Proposition 1, $\pi_x \in N(u_x), \pi_P \in N(u_P), \pi_y \in N(v_y), \pi_Q \in N(v_Q)$. Let $P = u_x \rightarrow P^1 \rightarrow P^0$ and let $Q = v_x \rightarrow Q^1 \rightarrow Q^0$. Then, $\{P, Q\}$ is a path partition of $AQ_{k+1}$ such that $P$ joins $u_x$ and $u_y$, $Q$ joins $v_x$ and $v_y$, and $|P| = |Q| = 2^k$. The construction in this case is shown in Fig. 5(a).

*Case 2: $u_x, u_y, v_x, v_y$ are in the same sub-augmented cube, and $v_x, v_y$ is in another sub-augmented cube.* Without loss of generality, assume that $u_x, u_y, v_x$ are in $AQ_k^0$ and that $v_y$ is in $AQ_k^1$. Let $x$ be a node in $AQ_k^0$ such that $x \notin \{u_x, u_y, u_z\}$ and $x \neq v_y$. By inductive hypothesis, there is a path partition $\{P^0, Q^0\}$ of $AQ_k^0$ such that $|P^0| = |Q^0|$, $\text{start}(P^0) = u_x$, $\text{end}(P^0) = u_y$, $\text{start}(Q^0) = v_x$, and $\text{end}(Q^0) = v_y$. Let $P^0 = u_x \rightarrow P$ and let $w = \text{start}(P^0)$. Consider that $\pi \notin \{\pi_x, \pi_y\}$. By $(\pi, \pi_x)$ and $(\pi, \pi_y)$ be two pairs of distinct nodes in $AQ_k^0$. By inductive hypothesis, there are two node-disjoint paths $P^1$ and $Q^1$ of $AQ_k^1$ such that $|P^1| = |Q^1| = 2^{k-1}$, $\text{start}(P^1) = \pi_x$, $\text{end}(P^1) = \pi_P$, $\text{start}(Q^1) = \pi_y$, and $\text{end}(Q^1) = \pi_Q$. By Proposition 1, $\pi_x \in N(u_x), \pi_P \in N(u_P), \pi_y \in N(v_y), \pi_Q \in N(v_Q)$. Let $P = u_x \rightarrow P^1 \rightarrow P^0$ and let $Q = v_x \rightarrow Q^1 \rightarrow Q^0$. Then, $\{P, Q\}$ is a path partition of $AQ_{k+1}$ such that $P$ joins $u_x$ and $u_y$, $Q$ joins $v_x$ and $v_y$, and $|P| = |Q| = 2^k$. The construction in this case is shown in Fig. 5(b). On the other hand, consider that $\pi \in \{\pi_x, \pi_y\}$. Since $|V(AQ_k^0)| = |V(AQ_k^1)| = 2^k \geq 4$, we can easily choose $w$ and $x$ such that $\pi \notin \{\pi_x, \pi_y\}$. Then, we can build two node-disjoint paths $P$ and $Q$ of $AQ_{k+1}$ by the same technique.

*Case 3: $u_x, u_y$ are in the same sub-augmented cube, and $v_x, v_y$ are in another sub-augmented cube.* Without loss of generality, assume that $u_x, u_y$ are in $AQ_k^0$ and that $v_x, v_y$ are in $AQ_k^1$. By Lemma 2, there are Hamiltonian paths $P$ and $Q$ of $AQ_k^0$ and $AQ_k^1$, respectively, such that $P$ joins $u_x$ and $u_y$ and $Q$ joins $v_x$ and $v_y$. Thus, $\{P, Q\}$ is a path partition of $AQ_{k+1}$ with $|P| = |Q| = 2^k$. Fig. 5(c) depicts the construction of the two paths in this case.

*Case 4: $u_x, v_x$ are in the same sub-augmented cube, and $u_y, v_y$ are in another sub-augmented cube.* Without loss of generality, assume that $u_x, v_x$ are in $AQ_k^0$ and that $u_y, v_y$ are in $AQ_k^1$. Let $x, y$ be two distinct nodes of $AQ_k^0$ such that $x, y \notin \{u_x, u_y\}$ and $x, y \notin \{v_x, v_y\}$. Let $(u_x, x)$ and $(v_y, y)$ be two pairs of distinct nodes in $AQ_k^0$ and let $(\pi, u_x)$ and $(\pi, v_y)$ be two pairs of distinct nodes in $AQ_k^1$. By inductive hypothesis, there are two node-disjoint paths $P^0$ and $Q^0$ of $AQ_k^0$ such that $|P^0| = |Q^0| = 2^{k-1}$, $\text{start}(P^0) = u_y$, $\text{end}(P^0) = x$, $\text{start}(Q^0) = v_x$, and $\text{end}(Q^0) = y$. In addition, there are two node-disjoint paths $P^1$ and $Q^1$ of $AQ_k^1$ such that $|P^1| = |Q^1| = 2^{k-1}$, $\text{start}(P^1) = \pi$, $\text{end}(P^1) = u_y$, $\text{start}(Q^1) = \pi$, and $\text{end}(Q^1) = v_y$. By Proposition 1, $\pi \in N(x)$ and $\pi \in N(y)$. Let $P = P^0 \Rightarrow P^1$ and let $Q = Q^0 \Rightarrow Q^1$. Then, $\{P, Q\}$ is a path partition of
Fig. 5. The constructions of two node-disjoint paths in $AQ_{k+1}$, with $k \geq 2$, for (a) $u_2, u_3, v_3, v_1 \in AQ_{k}$, (b) $u_2, u_3, v_3 \in AQ_{k}$ and $v_1 \in AQ_{k}$, (c) $u_2, u_1 \in AQ_{k}$ and $v_3, v_1 \in AQ_{k}$, and (d) $u_2, v_1 \in AQ_{k}$ and $u_3, v_3 \in AQ_{k}$, where dotted arrow lines indicate the paths, solid arrow lines indicate concatenated edges, and the symbol ‘×’ denotes the destruction to an edge in a path.

$AQ_{k+1}$ such that $P$ joins $u_2$ and $u_3$, $Q$ joins $v_3$ and $v_1$, and $|P| = |Q| = 2^k$. The construction in this case is shown in Fig. 5(d).

By the above cases, we have that $AQ_{k+1}$ contains two-equal path partition. By induction, $AQ_n$, with $n \geq 2$, contains two-equal path partition. Thus, we conclude the following theorem.

**Theorem 10.** For any integer $n \geq 2$, $AQ_n$ contains two-equal path partition.

V. CONCLUDING REMARKS

In this paper, we construct two edge-disjoint Hamiltonian cycles (paths) of a $n$-dimensional augmented cube $AQ_n$, for any integer $n \geq 3$. In addition, we prove that $AQ_n$, with $n \geq 2$, contains two-equal path partition. In the construction of two edge-disjoint Hamiltonian cycles (paths) of $AQ_n$, some edges are not used. It is interesting to see if there are more edge-disjoint Hamiltonian cycles of $AQ_n$ for $n \geq 4$. We would like to post it as an open problem to interested readers.

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