On the Embedding of Edge-Disjoint Hamiltonian Cycles in Transposition Networks

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Abstract—The presence of edge-disjoint Hamiltonian cycles provides an advantage when implementing algorithms that require a ring structure by allowing message traffic to be spread evenly across the network. Edge-disjoint Hamiltonian cycles also provide the edge-fault tolerant hamiltonicity of an interconnection network. We will study the property of edge-disjoint Hamiltonian cycles in transposition networks in this article. The networks under study belong to a subclass of Cayley graphs whose generators are subsets of all possible transpositions. The transposition networks include other famous network topologies as their subgraphs, such as meshes, hypercubes, star graphs, and bubble-sort graphs. In this paper, we first introduce a novel decomposition of transposition networks. Using the proposed decomposition, we construct three edge-disjoint Hamiltonian cycles in 4-dimensional transposition network. We then show that $n$-dimensional transposition network with $n \geq 5$ contains four edge-disjoint Hamiltonian cycles.

Keywords—Edge-disjoint Hamiltonian cycles; edge-fault tolerant hamiltonicity; transposition networks; interconnection networks

I. INTRODUCTION

Parallel computing is important for speeding up computation. The design of an interconnection network is the first thing to be considered. Many topologies have been proposed in the literature [3], [5], [7], [8], [10], [19], [20], [24], and the desirable properties of an interconnection network include symmetry, relatively small degree, small diameter, embedding capabilities, scalability, efficient routing, and fault-tolerant robustness. Cayley graphs arise naturally in interconnection networks and, hence, some interesting properties on them have been studied [4], [23], [26]. The transposition networks form a subclass of Cayley graphs and include other network topologies as their subgraphs, such as meshes, hypercubes, star graphs, and bubble-sort graphs [20], [29]. The transposition networks possess many attractive network properties, such as node symmetric [1], regular, sublogarithmic diameter, small fault-diameter, high node connectivity, embedding capabilities of meshes and hypercubes, simple shortest routing [20], lower bisection width [28], efficient node-disjoint path routing [9], and so on. In this article, we will investigate the property of edge-disjoint Hamiltonian cycles in transposition networks. The architecture of an interconnection network is usually modeled by a graph in which the nodes represent the processing elements and the edges represent the communication links. In this paper, we will use graph and network, vertex and node, and edge and link interchangeably.

A Hamiltonian cycle in a graph is a simple cycle that passes through every node of the graph exactly once. A graph is called Hamiltonian if it contains a Hamiltonian cycle. The ring structure is important for distributed computing, and its benefits can be found in [18]. Two Hamiltonian cycles in a graph are said to be edge-disjoint if there exists no common edge in them. A graph admits a Hamiltonian decomposition if all of its edges can be partitioned into disjoint Hamiltonian cycles. The edge-disjoint Hamiltonian cycles can provide an advantage for algorithms that make use of a ring structure [25]. Consider the problem of all-to-all broadcasting in which each node sends an identical message to all other nodes in the network. There is a simple solution for the problem using an $N$-node ring that requires $N - 1$ steps, i.e., at each step, every node receives a new message from its ring predecessor and passes the previous message to its ring successor. If the network admits edge-disjoint rings, then messages can be divided and the parts broadcast along different rings without any edge (link) contention. If the network can be decomposed into edge-disjoint Hamiltonian cycles, then the message traffic will be evenly distributed across all communication links. Edge-disjoint Hamiltonian cycles also form the basis of an efficient all-to-all broadcasting algorithm for networks that employ wormhole or cut-through routing [21]. Further, edge-disjoint Hamiltonian cycles also provide the edge-fault tolerant hamiltonicity of an interconnected network; that is, when a Hamiltonian cycle of an interconnected network contains one faulty edge, then the other edge-disjoint Hamiltonian cycle can be used to replace it for transmission. In this paper, we first construct three edge-disjoint Hamiltonian cycles in 4-dimensional transposition network, i.e., it admits a Hamiltonian decomposition. We then show that $n$-dimensional transposition network with $n \geq 5$ contains four edge-disjoint Hamiltonian cycles.

The $n$-dimensional transposition network, denoted by $TN_n$, was first introduced in [22] and briefly mentioned in [19] as the transposition graph. The $n$-dimensional transposition network $TN_n$ is a node-symmetric graph with $n!$ nodes, each node representing a distinct permutation of the set $\{1, 2, \cdots, n\}$. Let $p = p_1p_2\cdots p_n$ be a permutation of $\{1, 2, \cdots, n\}$ in which $p_i$ is called the $i$-th digit of $p$ for $1 \leq i \leq n$. The transposition of $p$ is a permutation obtained...
from \( p \) by swapping two digits in positions \( i \) and \( j \) of \( p \). Each transposition of node \( p \) in \( TN_n \) is a neighbor of \( p \). That is, two nodes are adjacent if and only if they differ by exactly one transposition \([6], [27]\). For example, node 1234 in \( TN_4 \) is adjacent to nodes 2134, 3214, 4321, 1324, 1432, 1423, 1243.

Transposition networks form a subclass of Cayley networks (graphs) \([1], [9]\) and include other network topologies as their subgraphs, such as meshes, hypercubes, star graphs, and bubble-sort graphs \([20], [29]\). An \( n \)-dimensional transposition network \( TN_n \) is a regular undirected graph with node degree \( \frac{n(n-1)}{2} \), has \( n! \) nodes and \( \frac{n(n-1)n}{2} \) edges \([20]\). In the literature, many interesting properties of transposition networks have been studied. The \( n \)-dimensional transposition network \( TN_n \) is always Hamiltonian \([6]\). Latifi et al. \([20]\) showed that the diameter of \( TN_n \) is \( n - 1 \), the fault diameter of \( TN_n \) is \( n \), the node connectivity of \( TN_n \) is \( \frac{n(n-1)}{2} \). \( TN_n \) is bipartite but does not contain any cycle of odd length, and that \( TN_n \) with \( n > 2 \) contains all cycles of even length \( \ell \) for \( 4 \leq \ell \leq n! \). Stacho et al. \([28]\) gave the lower and upper bounds on bisection width of transposition networks. Kalpakis et al. \([17]\) showed that the bisection width of \( TN_n \) equals \( \frac{n!}{n} \) when \( n \) is even. Suzuki et al. \([29]\) gave an \( O(n^3) \)-time algorithm to find a collection of node-disjoint paths connecting a source node and a set of destination nodes in an \( n \)-dimensional transposition network \( TN_n \). Fujita \([9]\) improved Suzuki et al.’s result to obtain an \( O(n^6) \)-time algorithm for the node-disjoint paths problem in \( TN_n \).

Related areas of investigation are summarized as follows. The edge-disjoint Hamiltonian cycles in \( k \)-ary \( n \)-cubes and hypercubes has been studied in \([2]\). Barth et al. \([3]\) showed that the butterfly networks contain two edge-disjoint Hamiltonian cycles. Petrovic et al. \([24]\) characterized the number of edge-disjoint Hamiltonian cycles in hyper-tournaments. Hsieh et al. \([11]\) constructed edge-disjoint spanning trees in locally twisted cubes. Hsieh et al. \([12]\) investigated the edge-fault tolerant hamiltonicity of an \( n \)-dimensional locally twisted cube. The existence of a Hamiltonian cycle in locally twisted cubes, twisted cubes, and crossed cubes has been shown \([13], [30], [31]\). In addition, the existence of a Hamiltonian cycle in transposition networks has been verified in \([6], [20]\). In \([14]\) and \([15]\), we presented linear time algorithms to construct two edge-disjoint Hamiltonian cycles in twisted cubes and augmented cubes. Hussak and Schröder \([16]\) showed that \( 5 \)-dimensional star graph admits a Hamiltonian decomposition, i.e., it contains two edge-disjoint Hamiltonian cycles. However, there has been little work reported so far on edge-disjoint Hamiltonian cycle property in transposition networks. In this paper, we will study the edge-disjoint Hamiltonian cycles property in transposition networks.

The rest of the paper is organized as follows. In Section II, we introduce some notations used in the paper and give a novel decomposition for transposition networks. Section III shows how to use our decomposition to construct edge-disjoint Hamiltonian cycles in transposition networks. Finally, we conclude this paper in Section IV.

II. PRELIMINARIES

We usually use a graph to represent the topology of an interconnection network. A graph \( G = (V, E) \) is a pair of the node set \( V \) and the edge set \( E \), where \( V \) is a finite set and \( E \) is a subset of \( \{(u, v) | (u, v) \) is an unordered pair of \( V \). We will use \( V(G) \) and \( E(G) \) to denote the node set and the edge set of \( G \), respectively. If \( u, v \) is an edge in a graph \( G \), we say that \( u \) is adjacent to \( v \) and \( u, v \) are incident to edge \((u, v) \). A neighbor of a node \( v \) in a graph \( G \) is any node that is adjacent to \( v \). Moreover, we use \( N_G(v) \) to denote the set of neighbors of \( v \) in \( G \). The subscript \( G \) of \( N_G(v) \) can be removed from the notation if it has no ambiguity.

Let \( G = (V, E) \) be a graph with node set \( V \) and edge set \( E \). A (simple) path \( P \) of length \( \ell \) in \( G \), denoted by \( v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_{\ell-1} \rightarrow v_\ell \), is a sequence \((v_0, v_1, \ldots, v_{\ell-1}, v_\ell) \) of nodes such that \((v_i, v_{i+1}) \in E \) for \( 0 \leq i \leq \ell - 1 \). The first node \( v_0 \) and the last node \( v_\ell \) visited by \( P \) are denoted by \( start(P) \) and \( end(P) \), respectively, and they are called the end nodes of \( P \). Path \( v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_{\ell-1} \rightarrow v_\ell \) is called the reversed path, denoted by rev(\( P \)), of path \( P \). That is, rev(\( P \)) visits the nodes of path \( P \) from \( end(P) \) to \( start(P) \) sequentially. In addition, \( P \) is a cycle if \(|V(P)| > 3 \) and \( end(P) \) is adjacent to \( start(P) \). A path \( P = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_{\ell-1} \rightarrow v_\ell \) may contain another subpath \( Q \), denoted as \( v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_i \rightarrow Q \rightarrow v_j \rightarrow \cdots \rightarrow v_{\ell-1} \rightarrow v_\ell \), where \( Q = v_i \rightarrow v_{i+1} \rightarrow \cdots \rightarrow v_j \) for \( 0 \leq i < j \leq \ell \). A path (or cycle) \( G \) is called a Hamiltonian path (or Hamiltonian cycle) if it contains every node of \( G \) exactly once. Two paths (or cycles) \( P_1 \) and \( P_2 \) connecting a node \( u \) to a node \( v \) are said to be edge-disjoint if and only if \( E(P_1) \cap E(P_2) = \emptyset \). Two paths (or cycles) \( Q_1 \) and \( Q_2 \) of graph \( G \) are called node-disjoint if and only if \( V(Q_1) \cap V(Q_2) = \emptyset \). Two node-disjoint paths \( Q_1 \) and \( Q_2 \) can be concatenated into a path, denoted by \( Q_1 \rightarrow Q_2 \), if \( end(Q_1) \) is adjacent to \( start(Q_2) \).

Now, we introduce transposition networks and give a novel decomposition on them. Transposition networks form a subclass of Cayley graphs (networks) and include other famous network topologies as their subgraphs such as hypercubes, star graphs, and bubble-sort graphs. Let \( V_n \) be the set of \( n! \) distinct permutations of the set \( \{1, 2, \ldots, n\} \). Let \( p = p_1 p_2 \cdots p_n \) be a permutation in \( V_n \), where \( p_i \) denotes the \( i \)-th digit of permutation \( p \). A transposition \( \varphi_{i,j} (1 \leq i < j \leq n) \) is a function that interchanges the \( i \)-th and \( j \)-th digits of a given permutation, which is formally defined as follows: for a given permutation \( p = p_1 p_2 \cdots p_n \) in \( V_n \),

\[
\varphi_{i,j}(p) = p_1 \overbrace{p_2 \cdots p_{i-1} p_{p_i+1} p_{p_i} p_{j+1} \cdots p_n}^{p_i \rightarrow p_{p_i}} \overbrace{p_{p_i+1} \cdots p_{j-1} p_j}^{p_j \rightarrow p_{j+1}}.
\]

For example, \( \varphi_{1,3}(1234) = 3214 \) and \( \varphi_{1,4}(1234) = 4231 \).

Next, we define the function \( \psi(p, p_i) \) to be the permutation obtained from a given permutation \( p = p_1 p_2 \cdots p_n \) by removing the digit \( p_i \). For instance, \( \psi(15234, 5) = 1234 \). We formally define the \( n \)-dimensional transposition network for \( n \geq 1 \) as follows.

**Definition 1.** \([9], [17]\) An \( n \)-dimensional transposition network \( TN_n \) on \( n \) digits, denoted by \( TN_n \), is an undirected graph with a node set equivalent to \( V_n \) and an edge set \( E_n \), where \( E_n = \{(p, \varphi_{i,j}(p)) | p \in V_n \text{ and } 1 \leq i < j \leq n\} \).

Fig. 1(a) depicts the \( 3 \)-dimensional transposition network \( TN_3 \). By the above definition, an \( n \)-dimensional transposition network \( TN_n \) is a regular graph with \( n! \) nodes and \( \frac{n(n-1)n}{2} \) edges. By analyzing the structure of transposition networks, we observe that an \( n \)-dimensional transposition
network $TN_n$ can be decomposed into $n$ sub-transposition networks $TN_{n-1}^1, TN_{n-1}^2, \cdots, TN_{n-1}^n$ such that $TN_{n-1}^i$, $1 \leq i \leq n$, contains any node $p$ with digit $n$ being the $i$-th digit of $p$. For each $i \in \{1, 2, \cdots, n\}$, $TN_{n-1}^i$ is isomorphic to $TN_{n-1}$. For example, Fig. 1(b) shows $TN_4$ containing four sub-transposition networks $TN_3^1, TN_3^2, TN_3^3, TN_3^4$. The above decomposition is first proposed in this article. Using the above decomposition, we show that $TN_n$, $n \geq 5$, contains four edge-disjoint Hamiltonian cycles in the next section.

![Fig. 1. (a) The 3-dimensional transposition network $TN_3$, and (b) the 4-dimensional transposition network $TN_4$ containing $TN_3^1, TN_3^2, TN_3^3, TN_3^4$, where $TN_3^i$, $4 \geq i \geq 1$, is isomorphic to $TN_3$ and contains all nodes $p$ with digit 4 being the $i$-th digit of $p$.](image)

### III. EDGE-DISJOINT HAMILTONIAN CYCLES IN TRANSPOSITION NETWORKS

In this section, we first show that 4-dimensional transposition network contains a Hamiltonian decomposition, i.e., there exist three edge-disjoint Hamiltonian cycles in $TN_4$. Then, we show that $n$-dimensional transposition network with $n \geq 5$ contains four edge-disjoint Hamiltonian cycles.

We first construct three edge-disjoint Hamiltonian cycles in 4-dimensional transposition network as follows. By using the concept of gray code, we construct three edge-disjoint Hamiltonian cycles of $TN_4$ in the following lemma.

**Lemma 1.** There are three edge-disjoint Hamiltonian paths (cycles) in 4-dimensional transposition network $TN_4$.

**Proof:** We prove this lemma by constructing three such paths $P_1$, $P_2$, and $P_3$. Let $P_1 = 1234 \rightarrow 2134 \rightarrow 2314 \rightarrow 2341 \rightarrow 2143 \rightarrow 1243 \rightarrow 1423 \rightarrow 2413 \rightarrow 2431 \rightarrow 4231 \rightarrow 4123 \rightarrow 4321 \rightarrow 4312 \rightarrow 1432 \rightarrow 1342 \rightarrow 1324 \rightarrow 1321 \rightarrow 1234 \rightarrow 1243 \rightarrow 2134 \rightarrow 2314 \rightarrow 2341 \rightarrow 2143 \rightarrow 1243 \rightarrow 1423 \rightarrow 2413 \rightarrow 2431 \rightarrow 4231 \rightarrow 4123 \rightarrow 4321 \rightarrow 4312 \rightarrow 1432 \rightarrow 1342 \rightarrow 1324 \rightarrow 1321 \rightarrow 1234 \rightarrow 1243 \rightarrow 2134 \rightarrow 2314 \rightarrow 2341 \rightarrow 2143 \rightarrow 1243 \rightarrow 1423 \rightarrow 2413 \rightarrow 2431 \rightarrow 4231 \rightarrow 4123 \rightarrow 4321 \rightarrow 4312 \rightarrow 1432 \rightarrow 1342 \rightarrow 1324 \rightarrow 1321 \rightarrow 1234 \rightarrow 1243 \rightarrow$.

Then, $P_1$, $P_2$, and $P_3$ form three edge-disjoint Hamiltonian paths of $TN_4$. Fig. 2 depicts the construction of these three Hamiltonian paths. Clearly, they form three edge-disjoint Hamiltonian cycles in $TN_4$.

![Fig. 2. Three edge-disjoint Hamiltonian paths (cycles) in $TN_4$, where solid arrow lines indicate a Hamiltonian path $P_1$, dashed arrow lines indicate the second edge-disjoint Hamiltonian path $P_2$, and dotted arrow lines depict the third edge-disjoint Hamiltonian path $P_3$.](image)

We then use the above lemma to construct four edge-disjoint Hamiltonian cycles in 5-dimensional transposition network $TN_5$. The construction is shown in the following lemma.

**Lemma 2.** The 5-dimensional transposition network $TN_5$ contains four edge-disjoint Hamiltonian cycles.

**Proof:** We first decompose $TN_5$ into 5 sub-transposition networks $TN_5^1, TN_5^2, TN_5^3, TN_5^4$, and $TN_5^5$, where $TN_5^4$, $5 \geq i \geq 1$, is isometric to 4-dimensional transposition network $TN_4$. By lemma 1, $TN_5^5, 5 \geq i \geq 1$, contains three edge-disjoint Hamiltonian cycles. For each $TN_5^i$ with $5 \geq i \geq 1$, let $P^5_i$, $P^5_{i'}$, and $P^5_{i''}$ be the three edge-disjoint Hamiltonian paths (cycles) of $TN_5^i$. We will select the end nodes of paths $P^5_i$, $P^5_{i'}$, and $P^5_{i''}$, $5 \geq i \geq 1$, to satisfy that $\psi(\text{start}(P^5_i), 5) = 1234$, $\psi(\text{end}(P^5_i), 5) = 3214$, $\psi(\text{start}(P^5_{i'}), 5) = 4321$, $\psi(\text{end}(P^5_{i'}), 5) = 3142$, and $\psi(\text{start}(P^5_{i''}), 5) = 3142$. Let $P^5_i = \text{start}(P^5_i) \rightarrow P^5_i$ for $3 \geq i \geq 1$ and $5 \geq i \geq 1$. We will construct three edge-disjoint Hamiltonian paths (cycles) of $TN_5$ as follows. For $3 \geq i \geq 1$, let $P_i = \text{start}(P^5_i) \rightarrow \text{start}(P^5_i) \rightarrow$.
We first make some sub-paths via these remainder edges. The paths via the edges not in $P_k$ start for $i \in \{1, 2, 3, 4, 5\}$, visits one node of $TN_i$ for $4 \geq i \geq 2$ exactly once, ends at one node of $TN_k$, and satisfies that $\psi(p, 5) = \psi(q, 5)$ for $p, q \in R$. These sub-paths, named $R_i$ for $15 \geq j \geq 1$, are shown in Fig. 4. Fig. 4 also depicts $Z_i$ for $i \in \{11, 12, 13, 21, 22, 23, 31, 32, 33\}$. Note that the nodes of all $Z_i$’s are the end nodes of paths $P^*_k$, and $P^*_i$ for $3 \geq i \geq 1$ and $5 \geq k \geq 1$. We then construct the fourth edge-disjoint Hamiltonian path (cycle) of $TN_5$ by using the above sub-paths.

**Lemma 3.** For any integer $n \geq 5$, there are four edge-disjoint Hamiltonian paths (cycles) in an $n$-dimensional transposition network $TN_n$.
Proof: We claim that there exist four edge-disjoint Hamiltonian paths (cycles) $P_1$, $P_2$, $P_3$, and $P_4$ in $T_{N_k}$ with $n > 5$ such that $P_τ = start(P_τ) → P_τ$ for $4 \geq τ \geq 1$ and $\{start(P_τ), start(P_τ), end(P_τ)\} \cap \{start(P_j), start(P_j), end(P_j)\} = \emptyset$ for $i \neq j$. Note that $end(P_τ) = end(P_τ)$ for any $τ$. We prove this claim by induction on $n$, the dimension of the transposition network. By Lemma 2, the claim holds true when $n = 5$. Assume that the claim is true for $n = k + 1$. We will prove that the claim holds true when $n = k + 1$. We first decompose $T_{N_k+1}$ into $k + 1$ sub-transposition networks $T_{N_k+1}^1, T_{N_k+1}^2, \ldots, T_{N_k+1}^k, T_{N_k+1}^{k+1}$, where for $1 \leq i \leq k + 1, T_{N_k+1}^i$ contains those nodes $p$ with digit $k + 1$ being the $i$-th digit of $p$. For each $i \in \{1, 2, \ldots, k + 1\}$, $T_{N_k+1}^i$ is isomorphic to $T_{N_k}$. By the induction hypothesis, there are four edge-disjoint Hamiltonian paths (cycles) $P_i^1$, $P_i^2$, $P_i^3$, and $P_i^4$ in $T_{N_k}^i$ for $k + 1 \geq i \geq 1$. Let $P_i^1 = start(P_i^1) → P_i^1$ for $4 \geq τ \geq 1$ and $k + 1 \geq i \geq 1$. By the induction hypothesis, $\{start(P_i^1), start(P_i^1), end(P_i^1)\} \cap \{start(P_j^1), start(P_j^1), end(P_j^1)\} = \emptyset$ for $i \neq j$. By the induction hypothesis, let $u_i$ and $v_i$ be the $j$-th nodes visited by paths $P_i^1$ and $P_i^1$, respectively, for $4 \geq τ \geq 1$, $k + 1 \geq i \geq 1$, and $2k + j \geq 1$. We may choose these above paths of $P_i^1$ such that $ψ(u_i, k + 1) = ψ(v_i, k + 1)$ for $2k + j \geq 1$, i.e., $u_j \in N(v_j)$. Consider the following two cases:

Case 1: $k$ is even. For $4 \geq τ \geq 1$, let $P_i = start(P_i^1) → start(P_i^2) → \cdots → start(P_i^k) → start(P_i^{k+1}) → P_i^{k+1} \Rightarrow$ rev($P_i^{k+1}$) $→ P_i^{k+1}$ $→ \cdots$. Let $rev(P_i^{k+1})$ be the reversed path of $P_i^{k+1}$ for $i \in \{2, 4, \ldots, k\}$. The construction of path $P_i$ is shown in Fig. 5(a). Since $P_i^1$, $P_i^2$, $P_i^3$, and $P_i^4$ for some $i$ are edge-disjoint, the constructed paths $P_1$, $P_2$, $P_3$, and $P_4$ form edge-disjoint Hamiltonian paths of $T_{N_k+1}$.

Case 2: $k$ is odd. By the same construction in Case 1, we can construct four edge-disjoint Hamiltonian paths of $T_{N_k+1}$ in this case below. For $4 \geq τ \geq 1$, let $P_i = start(P_i^1) → start(P_i^2) → \cdots → start(P_i^k) → start(P_i^{k+1}) → P_i^{k+1} \Rightarrow$ rev($P_i^{k+1}$) $→ P_i^{k+1}$ $→ \cdots$. Let $rev(P_i^{k+1})$ be the reversed path of $P_i^{k+1}$ for $i \in \{1, 3, \ldots, k\}$. The construction of path $P_i$ is depicted in Fig. 5(b). Since $P_i^1$, $P_i^2$, $P_i^3$, and $P_i^4$ for some $i$ are edge-disjoint, the constructed paths $P_1$, $P_2$, $P_3$, and $P_4$ are four edge-disjoint Hamiltonian paths of $T_{N_k+1}$.

Let $P_1$, $P_2$, $P_3$, and $P_4$ be the constructed edge-disjoint Hamiltonian paths in the above cases such that $P_τ = start(P_τ) → P_τ$ for $4 \geq τ \geq 1$. It is not difficult to see from the above constructions that $\{start(P_τ), start(P_τ), end(P_τ)\} \cap \{start(P_j), start(P_j), end(P_j)\} = \emptyset$ for $τ \neq j$. In addition, the edge $\{start(P_2), end(P_2)\}$ is different from the edge $\{start(P_2), end(P_2)\}$ for $τ \neq j$. Using the above, $P_1$, $P_2$, $P_3$, and $P_4$ form four edge-disjoint Hamiltonian cycles of $T_{N_k+1}$. Thus, the claim holds true when $n = k + 1$. By induction, the claim holds true and, hence, the lemma is true.

It immediately follows from Lemmas 1 and 3 that we conclude the following theorem.

Theorem 4. The 4-dimensional transposition network $T_{N_k}$ admits a Hamiltonian decomposition, and an n-dimensional transposition network $T_{N_n}$ with $n \geq 5$ contains four edge-disjoint Hamiltonian cycles.

In the proof of Lemma 2, we obtain an additional edge-disjoint Hamiltonian cycles in $T_{N_5}$ from three edge-disjoint Hamiltonian cycles of $T_{N_4}$. Whether the technique can be applied to the higher dimension of transposition networks remains open. Thus, we post the following conjecture as an open problem to interested readers.

Conjecture 5. The n-dimensional transposition network $T_{N_n}$ with $n \geq 5$ contains $(n - 1)$ edge-disjoint Hamiltonian cycles.

IV. CONCLUDING REMARKS

In this paper, we first introduce a novel decomposition of transposition networks. We then use the decomposition structure to show that the 4-dimensional transposition network $T_{N_4}$ admits a Hamiltonian decomposition and that an n-dimensional transposition network $T_{N_n}$ with $n \geq 5$ contains four edge-disjoint Hamiltonian cycles. An n-regular network is called Hamiltonian decomposable if it contains $\lfloor \frac{n-1}{2} \rfloor$ edge-disjoint Hamiltonian cycles. It would be interesting to see whether an n-dimensional transposition network $T_{N_n}$ with $n \geq 5$ is Hamiltonian decomposable. We would like to post it as an open problem to interested readers.

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