On an Unreliable Retrial Queue with General Repeated Attempts and $J$ Optional Vacations

Dong-Yuh Yang$^1$, Fu-Min Chang$^2$ and Jau-Chuan Ke$^3$

$^1$Institute of Information and Decision Sciences, National Taipei University of Business, Taipei, Taiwan

$^2$Department of Finance, Chaoyang University of Technology, Taichung, Taiwan

$^3$Department of Applied Statistics, National Taichung Institute of Technology, Taichung, Taiwan

Abstract:

This paper considers a single-server retrial queue with constant retrial rate and batch arrivals, where an unreliable server may take an additional optional vacation after the first essential vacation. Customers arrive at the system according to a compound Poisson process, and they are served immediately if the server is available; otherwise, they will enter a retrial orbit and form a single waiting line. If the orbit becomes empty, the server leaves for the first essential vacation. At the end of first essential vacation, the server may either remain idle in the system or take one of $J$ additional vacations. The server may break down at any instant while providing service. It is assumed that the service times and repair times of the server and the retrial times of customers are all arbitrarily distributed. By using the supplementary variable technique, we derive the probability generating functions of the system size at a random epoch and at a departure epoch. Various system performance measures and two reliability indices are developed. Finally, we deal with a cost optimization problem and provide numerical examples.

Keywords: $J$ optional vacations, random breakdown, retrial queue, stochastic decomposition, supplementary variable.
1. Introduction

In this paper, we consider a variant vacation policy for an M\(^{(k)}\)/G/1 retrial queue with general retrial times, where an unreliable server operates J optional vacations policy as soon as the system becomes empty. The vacation policy discussed in this paper is described as follows: when no customers are found in the orbit, the server goes on the first essential vacation. After the first essential vacation, he may either remain idle in the system or take one of Type j (j = 1, 2, ..., J) vacation. At an optional vacation completion epoch, the server waits for the customers (if any) in the orbit or new arriving customers. One such model has potential applications in the packet-switched networks, communication protocols, and so forth. A retrial queueing system is characterized by the feature that arriving customers who find the server busy leave the service area and join a retrial group, called orbit. The retrial queueing system plays an important role in the study of telephone switching systems, telecommunication networks and computer systems. In the past, there has been a considerable amount of work on the retrial queueing systems. We refer the readers to the survey paper by Yang and Templeton (1987), the books by Falin and Templeton (1997) and Artalejo and Gomez-Corral (2008), and the bibliography by Artalejo (1999a, 1999b, 2010) for more details.

In practical situations, we often encounter the case that the server is subject to breakdowns and repairs. For related work on the retrial queues with server breakdowns, Sherman and Kharoufeh (2006) considered an M/M/1 retrial queue with server breakdowns, infinite-capacity orbit and normal queue. Gharbi and Ioualalen (2006) applied the generalized stochastic petri nets (GSPNs) to analyze finite-source retrial systems with multiple unreliable servers. Wang et al. (2001) applied the supplementary variables method to obtain the explicit expressions of reliability indices of retrial queues with server breakdowns and repairs. Using the same technique, Choudhury and Deka (2008) generalized the classical M/G/1 retrial queue with server breakdowns by considering two phases of service. Wang (2006) derived both queueing and reliability measures of the M/G/1 retrial queues with general retrial times and server breakdowns. Gao et al. (2012) studied a repairable Geo/G/1 retrial queue with recurrent customers, Bernoulli feedback and general retrial times, where the server is subject to starting failures. They also showed that their model approximates the repairable continuous-time M/G/1 retrial queue with recurrent customers, Bernoulli feedback and general retrial times. The performance analysis of the M\(^{(k)}\)/G/1 retrial queue with server breakdowns, impatient customers and multi-optional services was presented by Bhagat and Jain (2013). Based on the embedded Markov chain technique and supplementary variable method, Gao and Wang (2014) obtained the performance and reliability measures the M/G/1 retrial queue with orbital search and non-persistent customers, where the server is subject to failure due to the negative arrivals. Moreover, a recent survey on the retrial queues with interruptions can also be found in Krishnamoorthy et al. (2014).
It is realistic that a server may become unavailable for a random period of time (called vacation time) when there are no customers in the system, in which the server performs an additional task during vacation times. Queueing systems with vacations have been analyzed by many researchers due to their wide range of applications, such as computer systems, communication networks, production/inventory systems, etc. A comprehensive and detailed review of the vacation models can be in Doshi (1986), Takagi (1991), Tian and Zhang (2006) and Ke et al. (2010). The steady state analysis of a single server retrial queue with batch arrivals, two phases of heterogeneous service and general vacation time under Bernoulli schedule was performed by Kumar and Arumuganathan (2008) using the supplementary variable technique. Chang and Ke (2009) considered an $M^{[k]}/G/1$ retrial queue with general retrial times, where the server leaves for at most $J$ vacations when no customers appear in the orbit. Maurya (2013) employed the maximum entropy principle to approximate the expected waiting time in the orbit of the $M^{[k]}/(G_1,G_2)/1$ retrial queue with two phases of service under Bernoulli vacation schedule. Choudhury and Ke (2014) dealt with the steady-state behavior of an $M/G/1$ retrial queue with server breakdowns, delayed repairs and general retrial times under Bernoulli vacation schedule. Other variations of retrial models with vacations can be referred to Wu and Yin (2011), Tao et al. (2012), Rajadurai et al. (2014) and many more. Recently, Gao and Wang (2015) applied the embedded Markov chain and supplementary variable technique to investigate the discrete-time queue $GI^V/GEO/I/N-G$ queue with randomized working vacations and at most $J$ vacations.

The rest of this paper is structured as follows. In the next section, we give a detailed description of the retrial queue under investigation. Section 3 constructs the mathematical model for the unreliable $M^{[k]}/G/1$ retrial queue with general retrial times under $J$ optional vacations policy. Moreover, the existence of the stochastic decomposition property of the system size distribution at a random epoch is also demonstrated. In Sections 4 and 5, we develop various system performance measures as well as two reliability indices. In section 6, we show that our model can be reduced to some existing models in the literature by an appropriate choice of parameter values. Section 7 is devoted to construct a cost model to determine the optimal value of the parameter $J$, so as to minimize the expected cost per unit time. Numerical experiments are given to examine the influence of the value of $J$ on the cost function and system performance measures for three vacation-time distributions in Section 8. In the final section, we draw conclusions and future research.

2. The Model

We consider an $M^{[k]}/G/1$ retrial queue with server breakdowns and general repeated attempts, where the unreliable server operates an optional vacations policy after the first essential vacation is completed. The detailed description of the model is given as follows:

- New customers arrive in batches according to a compound Poisson process with rate $\lambda$. Let $X_k$ denote the number of customers belonging to the $k$th arrival batch, where $X_k$, $k=1,2,3,...$, are with a common distribution
\[ \Pr[X_k = n] = \chi_n, \quad n = 1, 2, 3, \ldots. \]

Denote \( E[X(X-1)(X-k+1)], \; k \geq 1 \), the \( k \)th factorial moments of \( \{X_n\}_{n=1}^{\infty} \).

- There is a single server who provides service to all arriving customers. The service time of the server is an independently and identically distributed (i.i.d.) random variable \( S \) with distribution function \( S(t) \), Laplace-Stieltjes transform (LST) \( S(\theta) = E[e^{-\theta S}] \), and \( k \)th \( (k \geq 1) \) moment \( E[S^k] \).

- The server is subject to breakdowns at any time with a Poisson breakdown rate \( \alpha \) while providing service. As soon as a breakdown occurs, the server is sent for repair immediately, during which time he stops providing service to customers. The repair time of the server is an i.i.d. random variable \( D \) with distribution function \( D(t) \), LST \( D(\theta) = E[e^{-\theta D}] \), and \( k \)th \( (k \geq 1) \) moment \( E[D^k] \).

- As soon as the orbit is empty, the server leaves for the first essential vacation of random length \( V_0 \). When the server returns from the vacation, he may either wait idle for customers in the system with probability \( p_0 \) or take an additional optional vacation with probability \( p_j \), which is one of Type \( j \) \( (j = 1, 2, \ldots, J) \) vacation. At the end of an optional vacation, the server remains idle for the customer in the orbit or new arrivals. The vacation time of Type \( j \) vacation is assumed to be an i.i.d. random variable \( V_j \) with distribution function \( V_j(t) \), LST \( V_j(\theta) = E[e^{-\theta V_j}] \) and \( k \)th \( (k \geq 1) \) moment \( E[V_j^k] \), where \( j = 1, 2, \ldots, J \) and \( \sum_{j=0}^{J} p_j = 1 \). Similarly, the distribution function, LST and \( k \)th \( (k \geq 1) \) moment of the first essential vacation \( V_0 \) are denoted by \( V_0(t), V_0(\theta) = E[e^{-\theta V_0}] \) and \( E[V_0^k] \), respectively. Note that the server does not take any one of these optional vacations upon returning from the essential vacation when \( p_0 = 1 \). In this case, we mark \( J = 0 \).

Applying the concepts by Keilson and Servi (1986, 1989), the distribution of the total vacation times, denoted by \( V(t) \), for our retrial model is given by

\[ V(t) = p_0 V_0(t) + \sum_{j=1}^{J} p_j V_j(t), \]

where \( * \) is the symbol of convolution of two distribution functions.

- We assume that there is no waiting space and therefore if arriving customers find the server free, one of them begins his service; and the others leave the service area and join a pool of blocked customers called orbit. If arriving customers finding the server busy, on vacation, or broken down must leave the service area and are queued in the orbit in accordance with first come, first served discipline. Only the customer at the head of the orbit queue requests a service from the server. The inter-retrial time of the customer in the orbit is an i.i.d. random variable \( R \) with distribution function \( R(t) \), LST \( R(\theta) = E[e^{-\theta R}] \), and \( k \)th \( (k \geq 1) \) moment \( E[R^k] \).

- Various stochastic processes involved in this system are mutually independent of each others.
2.1. Practical justification of the model

The proposed model can be applied in the electronic mail (e-mail) system. The email is sent from the sender's mail server to the receiver's mail server via SMTP (simple mail transfer protocol). It provides a protocol for sending email between mail servers using a TCP (transmission control protocol) connection. Typically, a group of email messages arrive at the mail server in a Poisson stream. If the mail server is idle, one of the arriving email messages is selected to be processed, and the rest of the email messages will enter the buffer. After each processing job is completed, the mail server spends a random period of time to find the next email message from the buffer to process. The mail server is subject to breakdowns and repairs during the processing period. The virus scan is always performed when there are no mail messages in the buffer. When the virus scan is complete, the mail server remains idle to wait for new arriving email messages or performs other maintenance activities such as defragmentation, data compression, disk cleaning, and so on. In this scenario, the buffer in the sender's mail server, the receiver's mail server, the mail server searches for one email message from the buffer, the virus scan, and other maintenance activities correspond to the orbit, the server, the constant retrial policy, the essential vacation, and the optional vacations, respectively, in the queueing terminology. Consequently, this mailed system can be modeled as a batch arrival retrial queue with server breakdowns, constant retrial policy, and additional optional vacations.

3. The Analysis

We first develop the steady-state differential-difference equations for the $M^{[x]}/G/1$ retrial queue with server breakdowns, general retrial times, and $J$ optional vacations by treating the elapsed retrial time, elapsed service time, elapsed repair time, and elapsed vacation time as supplementary variables. Then, we derive the probability generating functions (PGFs) of the joint distribution for the server state and the number of customers in the system/orbit. Assume that the system is in a steady state condition. Let $O(t)$ be the number of customers in orbit, $R(t)$ be the elapsed retrial time, $S(t)$ be the elapsed service time, $D(t)$ be the elapsed repair time, $V_0(t)$ be the elapsed time of the first essential vacation, and $V_j(t)$ be the elapsed time of Type $j$ vacation. For further development of this retrial queueing model, let us define the random variable $\Delta(t)$ as follows:

$$\Delta(t) = \begin{cases} 
0, & \text{if the server is free at time } t, \\
1, & \text{if the server is busy at time } t, \\
2, & \text{if the server is under repair at time } t, \\
3, & \text{if the server is on the first essential vacation at time } t, \\
4, & \text{if the server is on Type 1 vacation at time } t, \\
\vdots & \\
j + 3, & \text{if the server is on Type } j \text{ vacation at time } t, \\
\vdots & \\
J + 3, & \text{if the server is on Type } J \text{ vacation at time } t. 
\end{cases}$$
Thus, the supplementary variables $R^{-}(t)$, $S^{-}(t)$, $D^{-}(t)$, and $V_{j}^{-}(t)$ are introduced to obtain a Markov process $\{O(t), \delta(t)\}$, where $\delta(t) = R^{-}(t)$ if $\Delta(t) = 0$ and $O(t) > 0$, $\delta(t) = S^{-}(t)$ if $\Delta(t) = 1$ and $O(t) \geq 0$, $\delta(t) = D^{-}(t)$ if $\Delta(t) = 2$ and $O(t) \geq 0$, $\delta(t) = V_{j}^{-}(t)$ if $\Delta(t) = j + 3$ and $O(t) \geq 0$ ($j = 0, 1, \ldots, J$). Now we define following limiting probabilities:

$$P_{n}(t) = \Pr\{O(t) = 0, \Delta(t) = 0\},$$

$$P_{n}(x,t)dx = \Pr\{O(t) = n, \delta(t) = R^{-}(t); x < R^{-}(t) \leq x + dx\}, \quad x > 0, \ n \geq 1,$$

$$\Pi_{n}(x,t)dx = \Pr\{O(t) = n, \delta(t) = S^{-}(t); x < S^{-}(t) \leq x + dx\}, \quad x > 0, \ n \geq 0,$$

$$Q_{n}(x,y,t)dx = \Pr\{O(t) = n, \delta(t) = D^{-}(t); y < D^{-}(t) \leq y + dy \mid S^{-}(t) = x\}, \quad (x, y) > 0, \ n \geq 0,$$

$$\Omega_{j,n}(x,t)dx = \Pr\{O(t) = n, \delta(t) = V_{j}^{-}(t); x < V_{j}^{-}(t) \leq x + dx\}, \quad x > 0, \ n \geq 0, \ 0 \leq j \leq J.$$

In steady-state, we set $P_{0} = \lim_{t \to \infty} P_{0}(t)$, and limiting densities $P_{n}(x) = \lim_{t \to \infty} P_{n}(x,t)$ for $x > 0$ and $n \geq 1$, $\Pi_{n}(x) = \lim_{t \to \infty} \Pi_{n}(x,t)$ for $x > 0$ and $n \geq 0$, $Q_{n}(x,y) = \lim_{t \to \infty} Q_{n}(x,y,t)$ for $(x, y) > 0$ and $n \geq 0$, and $\Omega_{j,n}(x) = \lim_{t \to \infty} \Omega_{j,n}(x,t)$ for $x > 0$ and $n \geq 0$. Further, it is assumed that $R(0) = 0$, $R(\infty) = 1$, $S(0) = 0$, $S(\infty) = 1$, $D(0) = 0$, $D(\infty) = 1$, $V_{j}(0) = 0$, $V_{j}(\infty) = 1$ ($j = 0, 1, 2, \ldots, J$); $R(x)$, $S(x)$, and $V_{j}(x)$ are continuous at $x = 0$; and $D(y)$ is continuous at $y = 0$, so that

$$\theta(x)dx = \frac{dR(x)}{1 - R(x)}; \quad \mu(x)dx = \frac{dS(x)}{1 - S(x)}; \quad \omega_{j}(x)dx = \frac{dV_{j}(x)}{1 - V_{j}(x)}; \quad \text{and} \quad \eta(y)dy = \frac{dD(y)}{1 - D(y)}$$

are the first order differential (hazard rate) functions of $R$, $S$, $V_{j}$ and $D$.

### 3.1. System size distribution at a random epoch

Following the arguments of Cox (1955), the Kolmogorov forward equations govern the system under the steady-state conditions can be written as follows:

\[
\lambda P_{0} = p_{0} \int_{0}^{\infty} \Omega_{0,0}(x) \omega_{0}(x) dx + \int_{0}^{\infty} \Omega_{1,0}(x) \omega_{1}(x) dx + \int_{0}^{\infty} \Omega_{2,0}(x) \omega_{2}(x) dx + \ldots
\]

\[
\int_{0}^{\infty} \Omega_{j,0}(x) \omega_{j}(x) dx,
\]

\[
\frac{d}{dx} P_{n}(x) + [\lambda + \theta(x)] P_{n}(x) = 0, \quad x > 0, \ n \geq 1,
\]

\[
\frac{d}{dx} \Pi_{n}(x) + [\lambda + \alpha + \mu(x)] \Pi_{n}(x) = \int_{0}^{\infty} \eta(y) Q_{n}(x,y) dy, \quad x > 0,
\]

\[
\frac{d}{dx} Q_{n}(x) + [\lambda + \alpha + \mu(x)] Q_{n}(x) = \lambda \sum_{k=1}^{n} \Delta_{k} \Pi_{k-1}(x) + \int_{0}^{\infty} \eta(y) Q_{n}(x,y) dy, \quad x > 0, \ n \geq 1,
\]

\[
\frac{d}{dy} Q_{0}(x,y) + [\lambda + \eta(y)] Q_{0}(x,y) = 0, \quad x > 0, \ y > 0,
\]

5
\[
\frac{d}{dy}Q_n(x, y) + [\lambda + \eta(y)]Q_n(x, y) = \lambda \sum_{k=1}^{n} \chi_k Q_{n-k}(x, y), \quad x > 0, \quad y > 0, \quad n \geq 1, \quad (6)
\]

\[
\frac{d}{dx} \Omega_{j,0}(x) + [\lambda + \omega_j(x)]\Omega_{j,0}(x) = 0, \quad x > 0, \quad 0 \leq j \leq J, \quad (7)
\]

\[
\frac{d}{dx} \Omega_{j,n}(x) + [\lambda + \omega_j(x)]\Omega_{j,n}(x) = \lambda \sum_{k=1}^{n} \chi_k \Omega_{j,n-k}(x), \quad x > 0, \quad n \geq 1, \quad 0 \leq j \leq J. \quad (8)
\]

These set of equations are to be solved under the following boundary conditions at \(x = 0:\)

\[
P_n(0) = p_0 \int_0^\infty \Omega_{0,n}(x) \omega_k(x) dx + \int_0^\infty \Omega_{1,n}(x) \omega_k(x) dx + \int_0^\infty \Omega_{2,n}(x) \omega_k(x) dx + \ldots + \int_0^\infty \Omega_{n,n}(x) \omega_k(x) dx, \quad n \geq 1,
\]

\[
\Pi_0(0) = \int_0^\infty P_1(x) \theta(x) dx + \lambda \chi_1 P_0,
\]

\[
\Pi_n(0) = \int_0^\infty P_{n+1}(x) \theta(x) dx + \lambda \sum_{k=1}^{n} \chi_k P_{n-k+1}(x) dx + \lambda \chi_{n+1} P_0, \quad n \geq 1,
\]

\[
\lambda P_0 = \int_0^\infty \Omega_{0,0}(x) \omega_k(x) dx,
\]

\[
\Omega_{0,n}(0) = \begin{cases} 
\int_0^\infty \Pi_0(x) \mu(x) dx, & n = 0, \\
0, & n \geq 1,
\end{cases}
\]

\[
\Omega_{j,n}(0) = p_j \int_0^\infty \Omega_{j,n}(x) \omega_k(x) dx, \quad n \geq 0 \quad \text{and} \quad 1 \leq j \leq J,
\]

and at \(y = 0\) and fixed values of \(x:\)

\[
Q_n(x, 0) = \alpha \Pi_n(x), \quad x > 0, \quad n \geq 0,
\]

and the normalization condition

\[
P_0 + \sum_{n=0}^{\infty} \int_0^\infty P_n(x) dx + \sum_{n=0}^{\infty} \left[ \int_0^\infty \Pi_n(x) dx + \int_0^\infty \int_0^\infty Q_n(x, y) dy dx \right] + \sum_{j=0}^{J} \sum_{n=0}^{\infty} \Omega_{j,n}(x) dx = 1.
\]

To solve the above equations, we define the following PGFs of \(\{\chi_n\}, \{P_n(\cdot)\}, \{\Pi_n(\cdot)\}, \{Q_n(\cdot)\}, \) and \(\{\Omega_{j,n}(\cdot)\}\) for \(|z| \leq 1:\)

\[
X(z) = \sum_{n=0}^{\infty} z^n \chi_n,
\]

\[
P(x; z) = \sum_{n=0}^{\infty} z^n P_n(x), \quad P(0; z) = \sum_{n=0}^{\infty} z^n P_n(0),
\]

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\[
\Pi(x; z) = \sum_{n=0}^{\infty} z^n \Pi_n(x), \quad \Pi(0; z) = \sum_{n=0}^{\infty} z^n \Pi_n(0),
\]
\[
Q(x, y; z) = \sum_{n=0}^{\infty} z^n Q_n(x, y), \quad Q(0, 0; z) = \sum_{n=0}^{\infty} z^n Q_n(x:)
\]
\[
\Omega_j(x; z) = \sum_{n=0}^{\infty} z^n \Omega_{j,n}(x), \quad \Omega_j(0; z) = \sum_{n=0}^{\infty} z^n \Omega_{j,n}(0), \quad 0 \leq j \leq J.
\]

For the limiting PGFs \( P(x; z), \Pi(x; z), Q(x, y; z) \) and \( \Omega_j(x; z) \), we define
\[
P(x; z) = \int_0^{\infty} P(x; z)dx, \quad \Pi(z) = \int_0^{\infty} \Pi(x; z)dx, \quad Q(z) = \int_0^{\infty} \int_0^{\infty} Q(x, y; z)dydx, \quad \text{and} \quad \Omega_j(z) = \int_0^{\infty} \Omega_j(x; z)dx.
\]

Multiplying Equation (2) by \( z^n \) and summing over \( n \) \((n = 1, 2, 3, \ldots)\), it finally yields
\[
\frac{\partial P(x; z)}{\partial x} + (\lambda + \theta(x))P(x; z) = 0, \quad x > 0.
\]

Similarly proceeding in the usual manner with Equations (3)-(15), we get
\[
\frac{\partial \Pi(x; z)}{\partial x} + [a(z) + \alpha + \mu(x)]\Pi(x; z) = \int_0^{\infty} \eta(y)Q_n(x, y; z)dy,
\]
\[
\frac{\partial Q(x, y; z)}{\partial y} + [a(z) + \eta(y)]Q(x, y; z) = 0,
\]
\[
\frac{\partial \Omega_j(x; z)}{\partial x} + [a(z) + \omega_j(x)]\Omega_j(x; z) = 0,
\]
\[
P(0; z) = p_0 \int_0^{\infty} \Omega_0(x; z)\omega_0(x)dx + \sum_{j=1}^{J} \int_0^{\infty} \Omega_j(x; z)\omega_j(x)dx + \int_0^{\infty} \Pi(x; z)\mu(x)dx
\]
\[
- \Omega_{0,0}(0) - \lambda P_0,
\]
\[
\Pi(0; z) = \frac{1}{z} \int_0^{\infty} P(x; z)\theta(x)dx + \frac{\lambda X(z)}{z} \left[ \int_0^{\infty} P(x; z)dx + P_0 \right],
\]
\[
\Omega_j(0; z) = p_j \int_0^{\infty} \Omega_0(x; z)\omega_0(x)dx, \quad 1 \leq j \leq J,
\]
\[
Q(x, 0; z) = a\Pi(x; z).
\]

where \( a(z) = \lambda (1 - X(z)) \) and \( x > 0 \).

Solving the partial differential Equations (17)-(20), we obtain
\[
P(x; z) = P(0; z)[1 - R(x)]e^{-\lambda x},
\]
\[ \Pi(x; z) = \Pi(0; z)[1 - S(x)]e^{-\alpha(z)x}, \quad (26) \]
\[ Q(x, y; z) = Q(x, 0; z)[1 - D(y)]e^{-\alpha(z)y}, \quad (27) \]
and
\[ \Omega_j(x; z) = \Omega_j(0; z)[1 - V_j(x)]e^{-\alpha(z)x}, \quad j = 0, 1, 2, \ldots, J, \quad (28) \]
where \( A(z) = a(z) + \alpha(1 - D(a(z))). \)

From Equation (7), we have
\[ \Omega_{0,0}(x) = \Omega_{0,0}(0)[1 - V_0(x)]e^{-\lambda x}. \quad (29) \]

Multiplying Equation (29) by \( a_0(x) \) on both sides and integrating with respect to \( x \) from 0 to \( \infty \), then from Equation (12) we have
\[ \Omega_{0,0}(0) = \frac{\lambda P_0}{V_0(\lambda)}. \quad (30) \]

From Equations (13) and (30), it shows that
\[ \Omega_0(0; z) = \frac{\lambda P_0}{V_0(\lambda)}. \quad (31) \]

It follows from Equations (23), (28) and (31) that
\[ \Omega_j(0; z) = p_j \frac{\lambda P_0}{V_0(\lambda)} V_0(a(z)), \quad 1 \leq j \leq J, \quad (32) \]

Integrating Equation (29) from 0 to \( \infty \) and using Equation (30) again, we get
\[ \Omega_{0,0} = \frac{P_0[1 - V_0(\lambda)]}{V_0(\lambda)}. \quad (33) \]

It is noted that \( \Omega_{0,0} \) represents the steady-state probability that no customers are present in the system while the server is on the first essential vacation.

Inserting Equation (25) in Equation (22) and after some calculation, we obtain
\[ \Pi(0; z) = P(0; z) \left[ \frac{R(\lambda) + X(z)(1 - R(\lambda))}{z} \right] + \frac{\lambda X(z)}{z} P_0. \quad (34) \]

Substitution of Equations (26), (28) and (30)-(32) into Equation (21) gives
\[ P(0; z) = \frac{\lambda P_0}{V_0(\lambda)} \left[ V_0(a(z))[p_0 + \sum_{j=1}^{J} p_j V_j(a(z))] - 1 \right] + \Pi(0; z) S(A(z)) - \lambda P_0. \quad (35) \]

Solving \( P(0; z) \) from Equation (34)-(35), it finally yields after some simplification
\[ P(0; z) = \frac{\lambda P_0 \left\{ \frac{z}{V_0(\lambda)} \left[ V_0(a(z))[p_0 + \sum_{j=1}^{J} p_j V_j(a(z))] - 1 \right] + X(z)S(A(z)) - z \right\}}{z - S(A(z))[R(\lambda) + X(z)(1 - R(\lambda))]}. \] (36)

One can obtain \( P(x; z) \) from Equations (25) and (36) that
\[ P(x; z) = \frac{\lambda P_0 \left\{ \frac{z}{V_0(\lambda)} \left[ V_0(a(z))[p_0 + \sum_{j=1}^{J} p_j V_j(a(z))] - 1 \right] + X(z)S(A(z)) - z \right\}}{z - S(A(z))[R(\lambda) + X(z)(1 - R(\lambda))]}, \] (37)

which leads to
\[ P(z) = \int_0^\infty P(x; z)dx \]
\[ P_0[1 - R(\lambda)] \left\{ \frac{z}{V_0(\lambda)} \left[ V_0(a(z))[p_0 + \sum_{j=1}^{J} p_j V_j(a(z))] - 1 \right] + X(z)S(A(z)) - z \right\}, \] (38)

From Equations (26), (34) and (36), we have
\[ \Pi(x; z) = \frac{\lambda P_0 \left\{ \frac{z}{V_0(\lambda)} \left[ V_0(a(z))[p_0 + \sum_{j=1}^{J} p_j V_j(a(z))] - 1 \right] + X(z)S(A(z)) - z \right\}}{z - S(A(z))[R(\lambda) + X(z)(1 - R(\lambda))]}
\[ \times \left[ \frac{R(\lambda) + X(z)(1 - R(\lambda))}{z} + \frac{\lambda X(z)}{z} - P_0 \right] \times [1 - S(x)e^{-\lambda x}]. \] (39)

It follows from Equation (39) that
\[ \Pi(z) = \int_0^\infty \Pi(x; z)dx \]
\[ \lambda P_0 \left\{ \frac{V_0(a(z))[p_0 + \sum_{j=1}^{J} p_j V_j(a(z))] - 1}{V_0(\lambda)} - 1 \right\} \left[ R(\lambda) + X(z)(1 - R(\lambda)) \right] + X(z) \]
\[ \frac{z - S(A(z))[R(\lambda) + X(z)(1 - R(\lambda))]}{z - S(A(z))[R(\lambda) + X(z)(1 - R(\lambda))]} \]
\[ \times \frac{1 - S(A(z))}{A(z)}. \] (40)

From Equations (28), (31) and (32), it yields
\[ \Omega_0(x; z) = \frac{\lambda P_0}{V_0(\lambda)} [1 - V_0(x)] e^{-a(z)x}, \quad (41) \]

\[ \Omega_j(x; z) = p_j \frac{\lambda P_0}{V_0(\lambda)} V_0(a(z))[1 - V_j(x)] e^{-a(z)x}, \quad j = 1, 2, ..., J. \quad (42) \]

Integrating Equations (41)-(42) with respect to \( x \) from 0 to \( \infty \), we have

\[ \Omega_0(z) = \frac{\lambda P_0 [1 - V_0(a(z))]}{V_0(\lambda) a(z)}, \quad (43) \]

\[ \Omega_j(z) = \frac{p_j P_0 V_0(a(z))[1 - V_j(a(z))]}{V_0(\lambda) [1 - X(z)]}, \quad j = 1, 2, ..., J. \quad (44) \]

Inserting Equation (24) into Equation (27), \( Q(x, y; z) \) can be expressed as

\[ Q(x, y; z) = \alpha \Pi(x; z)[1 - D(y)] e^{-a(z)y}, \quad (45) \]

where \( \Pi(x; z) \) is given in Equation (39).

Calculating the double integral \( \int_0^\infty \int_0^\infty Q(x, y; z) dx\,dy \), we obtain

\[ Q(z) = \Pi(z) \frac{\alpha [1 - D(a(z))]}{a(z)}, \quad (46) \]

where \( \Pi(z) \) is given in Equation (40).

Finally, the unknown constant \( P_0 \) can be determined by using the normalization condition (16), which is equivalent to \( P_0 + P(1) + \Pi(1) + Q(1) + \sum_{j=0}^{J} \Omega_j(1) = 1 \). Thus, we obtain

\[ P_0 = \frac{1 - \rho_H - E[X](1 - R(\lambda))}{\lambda E(V_0) + \lambda \sum_{j=1}^{J} p_j E(V_j) + R(\lambda)}, \quad (47) \]

where \( \rho_H = \lambda E[X]E[S](1 + \alpha E[D]) \). From Equation (47), the necessary and sufficient condition for the system to be stable is given by \( \rho_H + E[X](1 - R(\lambda)) < 1 \).

Let \( \Phi(z) = P_0 + P(z) + z\Pi(z) + zQ(z) + \sum_{j=0}^{J} \Omega_j(z) \) be the PGF of the system size distribution at a random epoch. Then, we have
\[
\Phi_s(z) = P_0 \frac{V_0(a(z)) \left( p_0 + \sum_{j=1}^{\infty} p_j V_j(a(z)) \right) - 1}{V_0(\lambda)} \left[ R(\lambda) + X(z)(1 - R(\lambda)) \right] + R(\lambda)(X(z) - 1) \\
\times \frac{S(A(z))(z - 1)}{X(z) - 1},
\]

where \( P_0 \) is given in Equation (47).

3.2. Stochastic decomposition

Stochastic decomposition can be found extensively in many studies concerning M/G/1 type queueing models with server vacations (e.g., see Doshi, 1986, Takagi, 1991, Fuhrman and Cooper, 1985). In this subsection, we show the existence of the stochastic decomposition property for the M^{[x]}/G/1 retrial queue with server breakdowns, general retrial times and \( J \) optional vacations. From Equation (48), the PGF of the stationary system size distribution can be decomposed into two independent terms:

\[
\Phi_s(z) = \xi(z) \times \phi_{M^{[x]}_{v1}/G/1}(z),
\]

where

\[
\xi(z) = P_0[z - S(A(z))]
\]

\[
\left[\frac{V_0(a(z)) \left( p_0 + \sum_{j=1}^{\infty} p_j V_j(a(z)) \right) - 1}{V_0(\lambda)} \left[ R(\lambda) + X(z)(1 - R(\lambda)) \right] + R(\lambda)(X(z) - 1) \right] \frac{(1 - \rho_H)(X(z) - 1)}{z - S(A(z))} \left[ R(\lambda) + X(z)(1 - R(\lambda)) \right] + R(\lambda)(X(z) - 1)
\]

and

\[
\phi_{M^{[x]}_{v1}/G/1}(z) = \frac{(1 - \rho_H)(z - 1)S(A(z))}{z - S(A(z))}.
\]

We observe from Equation (49) that the stationary system size distribution of the M^{[x]}/G/1 retrial queue with server breakdowns, general repeated attempts and \( J \) optional vacations is the convolution of two independent random variables: one (the first term) is the PGF of the number of customers in the variant vacation system at a random point in time given that the server is on vacation or idle, and the other (the second term) is the PGF of the stationary system size distribution of the ordinary M^{[x]}/G/1 queueing system with server breakdowns.

Thus, it confirms that the decomposition result of Fuhrman and Cooper (1985) also holds for this variant vacation M^{[x]}/G/1 retrial queue.
3.3. System size distribution at a departure epoch

To obtain the system size distribution at a departure epoch, we first denote \( \{ \pi_j; j \geq 0 \} \) as the probability that a departing customer sees \( j \) customers in the system. Following the Poisson Arrivals See Time Averages (PASTA) property (see Wolff, 1982), it is stated that a departing customer will see \( j \) customers in the system just after a departure if and only if there were \( (j + 1) \) customers in the system just before the departure. Thus, we can write

\[
\pi_j = K_0 \int_0^{\infty} \Pi_j(x) \mu(x) dx, \quad j \geq 0,
\]

where \( K_0 \) is the normalizing constant.

Let \( \pi(z) = \sum_{j=0}^{\infty} z^j \pi_j \) be the PGF of the number of customers at a departure epoch. Multiplying Equation (50) by \( z^j \) and then summing from \( j = 0 \) to \( j = \infty \), we get

\[
\pi(z) = K_0 \lambda P_0 S(A(z)) z - S(A(z))[R(\lambda) + X(z)(1 - R(\lambda))].
\]

Applying the normalizing condition \( \pi(1) = 1 \), it follows that

\[
K_0 = \frac{1}{\lambda E(X)}.
\]

Inserting Equation (52) into Equation (51), we get

\[
\pi(z) = P_0 S(A(z)) z - S(A(z))[R(\lambda) + X(z)(1 - R(\lambda))]/E(X)(z - S(A(z))[R(\lambda) + X(z)(1 - R(\lambda))]).
\]

From Equation (53), it is apparent that \( \pi(z) \) can be decomposed into two independent terms

\[
\pi(z) = \frac{1 - X(z)}{E[X](1 - z)} \times \Phi_\xi(z) = \Delta(z) \times \xi(z) \times \phi_{M^{[X]}G/1}(z).
\]

Equation (54) shows the relationship between the system size distribution at a random epoch and the system size distribution at a departure epoch for the M\(^X\)/G/1 retrial queue with server breakdowns, general retrial times and \( J \) optional vacations. It is worth

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mentioning that \( \pi(z) \) can be decomposed into three independent random variables: (i) \( \Delta(z) \) is the number of customers placed before a tagged customer in a batch in which the tagged customer arrives; and (ii) \( \xi(z) \) and \( \phi_{N_1}(z) \) have been mentioned in the previous section.

4. System Performance Measures

In this section, we aim at obtaining the explicit expressions of some important system performance measures of the \( M^{[\ell]} / G / 1 \) retrial queue with server breakdowns, general repeated attempts and \( J \) optional vacations. First of all, let us define the probabilities of the server state as follows:

\[
P_b = \text{the probability that the server is busy};
\]

\[
P_i = \text{the probability that the server is idle during the retrial time};
\]

\[
P_v = \text{the probability that the server is on vacation};
\]

\[
P_d = \text{the probability that the server is broken down}.
\]

From Equations (38), (40), (43)-(44) and (46)-(47), we obtain

\[
P_b = \lambda E[X] E[S], \quad (55)
\]

\[
P_i = \frac{[1 - R(\lambda)] \left[ \lambda E[X] \left( E(V_0) + \sum_{j=1}^{J} p_j E(V_j) \right) - V_0(\lambda) \left[ 1 - E[X] (1 + \lambda E[S] (1 + \alpha E[D])) \right] \right]}{\lambda E(V_0) + \lambda \sum_{j=1}^{J} p_j E(V_j) + V_0(\lambda) R(\lambda)}, \quad (56)
\]

\[
P_v = \frac{\left( \lambda E(V_0) + \lambda \sum_{j=1}^{J} p_j E(V_j) \right) \left[ 1 - \rho_H - E[X] (1 - R(\lambda)) \right]}{\lambda E(V_0) + \lambda \sum_{j=1}^{J} p_j E(V_j) + V_0(\lambda) R(\lambda)}, \quad (57)
\]

\[
P_d = \alpha \lambda E[X] E[S] E[D]. \quad (58)
\]

Under the steady state condition, differentiating Equation (48) with respect to \( z \) at the point \( z = 1 \), and we can obtain the expected number of customers in the system at random epoch \( L_z \) as follows:
\[ L_s = \frac{\lambda^2 E[X] \left( E[V_0] + \sum_{j=1}^{J} p_j E[V_j^2] + 2E[V_0] \sum_{j=1}^{J} p_j E[V_j] \right)}{2 \left( \lambda E[V_0] + \lambda \sum_{j=1}^{J} p_j E[V_j] + V_0(\lambda) R(\lambda) \right)} \]

\[ + \frac{\lambda E[X] \left( E[V_0] + \sum_{j=1}^{J} p_j E[V_j] \right) (1 - R(\lambda))}{\lambda E[V_0] + \lambda \sum_{j=1}^{J} p_j E[V_j] + V_0(\lambda) R(\lambda)} \]

\[ + \frac{(\lambda E[X])^2 \left( E[S^2](1 + \alpha E[D])^2 + \alpha E[S] E[D^2] \right)}{2 \left( 1 - E[X](1 - R(\lambda)) - \rho_\eta \right)} \]

Finally, we derive the expected length of the busy cycle, denoted as \( E[T_c] \), under the steady state condition. Applying the argument of alternating renewal process, it leads to the well-known result as follows:

\[ E[T_c] = \frac{1}{\lambda E[X] \pi_0}. \]
where \( \pi_0 \) can be obtained by putting \( z=0 \) in Equation (53). Substituting \( \pi_0 \) into Equation (64) gives

\[
E[T_e] = \frac{\lambda E(V_0) + \lambda \sum_{j=0}^{J} p_j E(V_j) + V_0(\lambda)R(\lambda)}{\lambda \left[ V_0(\lambda) - V_0(\lambda)[p_0 + \sum_{j=1}^{J} p_j V_j(\lambda)] + 1 \right] \left[ 1 - E[X](1 - R(\lambda)) - \rho_H \right]}.
\]

(65)

5. Reliability Indices

Under the steady-state condition, we develop two reliability indices of the \( M^{[s]}/G/1 \) retrial queue with server breakdowns, general repeated attempts and \( J \) optional vacations. One is the system availability, and another is the failure frequency. Assume that \( A_e \) is the steady-state availability of the server, which is the probability that the server is either serving a customer or in an idle period. We derive \( A_e \) by the following equation:

\[
A_e = P_0 + \int_0^\infty \Pi(x;1).
\]

Let us define \( M_f \) as the steady-state failure frequency of the server, which can be computed as follows:

\[
M_f = \alpha \int_0^\infty \Pi(x;1).
\]

Using Equations (40) and (47), we get

\[
A_e = \frac{1 - E[X] \left[ (1 - R(\lambda)) + \lambda E[S] \left[ 1 + \alpha E[D] - \frac{\lambda E(V_0) + \lambda \sum_{j=1}^{J} p_j E(V_j)}{V_0(\lambda)} - R(\lambda) \right] \right]}{\frac{\lambda E(V_0) + \lambda \sum_{j=1}^{J} p_j E(V_j)}{V_0(\lambda)} + R(\lambda)}
\]

(66)

and

\[
M_f = \alpha \lambda E[X] E[S].
\]

(67)

6. Special Cases

We discuss some special cases of the \( M^{[s]}/G/1 \) retrial queue with server breakdowns, general repeated attempts and \( J \) optional vacations, which are consistent with existing results in the literature.

Case 1: Suppose that \( p_0 = 1 \) and \( \alpha = 0 \), our model is reduced to the \( M^{[s]}/G/1 \) retrial queue with general retrial times and single vacation. In this case, Equation (48) for \( \Phi_s(z) \) and Equation (61) for \( L_q \) can be written as follows:
\[
\Phi_s(z) = \frac{1 - E[X]\left(1 - R(\lambda) - \lambda E[S]\right)}{\frac{\lambda E(V_0)}{V_0(\lambda)} + R(\lambda)} \times \frac{\left[V_0(\lambda - \lambda X(z)) - 1\right]}{\lambda E(V_0)} \times \frac{\left[R(\lambda) + X(z)(1 - R(\lambda))\right]}{\lambda E(X(z - 1))} + R(\lambda)(X(z) - 1)
\]
\[
\times \frac{S(\lambda - \lambda X(z))(z - 1)}{X(z) - 1},
\]

(68)

and

\[
L_q = \frac{\lambda E[X]\left(\lambda E[V_0^2] + 2E[V_0]\left(1 - R(\lambda)\right)\right)}{2\left(\lambda E[V_0] + V_0(\lambda)R(\lambda)\right)} + \frac{\left(\lambda E[X]\right)^2 E[S^2]}{2\left(1 - E[X](1 - R(\lambda)) - \rho_H\right)}
\]
\[
+ \frac{\lambda E[X]\left(E[S](1 - R(\lambda)) + \lambda E[X]E[S]\right)}{1 - E[X](1 - R(\lambda)) - \lambda E[X]E[S]} \times \frac{\lambda E[X(1 - R(\lambda) + E[S])]}{2\left(1 - E[X](1 - R(\lambda)) - \lambda E[X]E[S]\right)},
\]

(69)

which are in accordance with those of Chang and Ke's system (2009) with \( J = 1 \). Note that for \( \Pr[X = 1] = 1 \), the above equations (68)-(69) are coincided with the results of Kumar and Arivudainambi (2002) while \( p = 1 \).

Case 2: If \( R(\lambda) \rightarrow 1 \) and \( p_0 = 1 \), our model describes the ordinary M\(^{[s]}\)/G/1 queue with an unreliable server and single vacation. In this case, Equation (48) for \( \Phi_s(z) \) and Equation (63) for \( W_q \) can be shown as follows:

\[
\Phi_s(z) = \frac{1 - \rho_H}{\lambda E(V_0) + V_0(\lambda)} \times \frac{\left(V_0(a(z)) - 1\right) + V_0(\lambda)(X(z) - 1)}{z - S(A(z))} \times \frac{S(A(z))(z - 1)}{X(z) - 1}
\]

(70)

and

\[
W_q = \frac{\lambda E[V_0^2]}{2\left(\lambda E[V_0] + V_0(\lambda)\right)} + \frac{\lambda E[X]\left(E[S^2](1 + \alpha E[D]) + \alpha E[S]E[D^2]\right)}{2\left(1 - \rho_H\right)}
\]
\[
+ \frac{E[X(1 - R(\lambda))]E[S](1 + \alpha E[D])}{2E[X(1 - \rho_H)]}.
\]

(71)

The above results are in agreement with the results in Section 3.2 of Ke and Huang (2012) when \( p = 1 \) and \( D'(\theta) = 0 \). Note that for \( \alpha = 0 \), our model can be further reduced to the ordinary M\(^{[s]}\)/G/1 queue with single vacation. Then, Equation (70) for \( \Phi_s(z) \) and Equation (71) for \( W_q \) are identical to the results found by Ke and Chu (2006) if \( J = 1 \).

Case 3: When \( \Pr[X = 1] = 1 \), \( p_0 = 1 \) and \( \Pr[V_0 = 0] = 1 \), our model becomes the M/G/1 retrial queue with an unreliable server and general retrial times. In this case, Equation (48) for \( \Phi_s(z) \) and Equation (61) for \( L_q \) can be simplified to the following expressions:
\[
\Phi_s(z) = \frac{(R(\lambda) - \rho_H)(1-z)S(A(z))}{S(A(z))R(\lambda)(1-z) - z[1-S(A(z))]} 
\] (72)

and
\[
L_q = \frac{\lambda^2 \left[ E[S^2]{(1+\alpha E[D])^2} + \alpha E[S]E[D^2] \right]}{2 \left( R(\lambda) - \rho_H \right)} + \frac{\rho_H (1-R(\lambda))}{R(\lambda) - \rho_H}, 
\] (73)

which are consistent with the queue without Bernoulli feedback mechanism and delaying repair in Choudhury and Ke (2014). Note that for \( \alpha = 0 \), Equation (72) for \( \Phi_s(z) \) is in agreement with Equation (16) in Gomes-Corral (1999).

7. **Optimal Control**

We construct the expected cost function per unit time for the unreliable M^{[x]}/G/1 retrial queue with \( J \) optional vacations policy. To make the analysis more tractable, it is assumed that the retrial time \( R \) is exponentially distributed with LST \( \tilde{R}(\theta) = \frac{\gamma}{\theta^2} \), where \( \gamma \) is the mean retrial rate. Our objective is to determine the optimal number of optional vacations \( J \), say \( J^* \), so as to minimize this cost function. Let

\[
\begin{align*}
C_w & = \text{the waiting cost per unit time for each customer present in the orbit;} \\
C_s & = \text{the setup cost per busy cycle.}
\end{align*}
\]

Based on the renewal reward theory (see Ross, 1983), the expected cost function per unit time is given by
\[
F(J) = C_u L_q + \frac{C_s}{E[T_1]}. 
\] (74)

Substituting Equations (61) and (65) into Equation (74), it yields
\[
F(J) = C_u \lambda^2 (\lambda + \gamma) E[X] \left[ E[V_0] + \sum_{j=1}^{J} p_j E[V_j] + 2E[V_0] \sum_{j=1}^{J} p_j E[V_j] \right] \\
\quad \times \frac{2 \left( \lambda (\lambda + \gamma) E[V_0] + \sum_{j=1}^{J} p_j E[V_j] \right) + \gamma V_0(\lambda)}{\lambda E[V_0] + \lambda \sum_{j=1}^{J} p_j E[V_j] \left( \lambda + \gamma \right) + \gamma V_0(\lambda)} \\
\quad + \frac{C_u \lambda^2 E[X] \left[ E[V_0] + \sum_{j=1}^{J} p_j E[V_j] \right]}{\lambda E[V_0] + \lambda \sum_{j=1}^{J} p_j E[V_j] \left( \lambda + \gamma \right) + \gamma V_0(\lambda)} \\
\quad + \frac{\lambda \left[ V_0(k) - V_0(k) p_0 + \sum_{j=1}^{J} p_j V_j(k) \right] + (1 - \rho_H)(\lambda + \gamma) - \lambda E[X]}{\left( E(V_0) + \sum_{j=1}^{J} p_j E[V_j] \right) (\lambda + \gamma) + \gamma V_0(\lambda)} + C_u K, 
\] (75)

where
\[ K = \frac{(\lambda E[X])^2(\lambda + \gamma)\left(E[S^2](1 + \alpha E[D])^2 + \alpha E[S]E[D^2]\right) + 2\lambda E[X] \rho_\mu + E[X(X-1)](\lambda^2 + \lambda E[S](1 + \alpha E[D])(\lambda + \gamma))}{2[(1 - \rho_\mu)(\lambda + \gamma) - \lambda E[X]]} \]

is independent of \( J \).

In order to find the minimum of \( F(J) \), we first investigate the behavior of \( F(J) \). Let

\[
\begin{align*}
A_1 &= 2\lambda(\lambda + \gamma)E[V_0] + 2\lambda V_0(\lambda) \quad \text{,} \quad A_2 = 2\lambda(\lambda + \gamma) \quad \text{,} \quad A_3 = C_\alpha \lambda^2(\lambda + \gamma)E[X]E[V_0^2] + 2C_\alpha \lambda^2 E[X]E[V_0^2] \quad \text{,} \\
A_4 &= C_\alpha \lambda^2(\lambda + \gamma)E[X] \quad \text{,} \quad A_5 = 2C_\alpha \lambda^2 E[X]\left(\lambda + \gamma)E[V_0^2]\right) + 1 \quad \text{,} \\
A_6 &= 2\lambda(V_0(\lambda) - V_0(\lambda)p_0 + 1)((1 - \rho_\mu)(\lambda + \gamma) - \lambda E[X]) \quad \text{,} \quad A_7 = -2\lambda V_0(\lambda)((1 - \rho_\mu)(\lambda + \gamma) - \lambda E[X]) \quad \text{,} \\
f(J) &= \sum_{j=1}^{j'} p_j E[V_j^2] \quad \text{,} \quad h(J) = \sum_{j=1}^{j'} p_j E[V_j] \quad \text{,} \quad g(J) = \sum_{j=1}^{j'} p_j g_j(\lambda) \quad \text{. It follows that}
\end{align*}
\]

\[ F(J) - F(J + 1) = \frac{N_{1,j} + C_n N_{2,j}}{D_j} - \frac{N_{1,j+1} + C_n N_{2,j+1}}{D_{j+1}} = \frac{(N_{1,j}D_{j+1} - N_{1,j+1}D_j) + C_n(N_{2,j}D_{j+1} - N_{2,j+1}D_j)}{D_j D_{j+1}} = \frac{X(J) - Y(J)}{D_j D_{j+1}}, \tag{76} \]

where

\[
\begin{align*}
D_j &= A_1 + A_2 f(J) \quad \text{,} \quad N_{1,j} = A_3 + A_4 h(J) + A_5 f(J) \quad \text{,} \quad N_{2,j} = A_6 + A_7 g(J) \quad \text{,} \\
X(J) &= N_{1,j}D_{j+1} - N_{1,j+1}D_j \quad \text{,} \quad Y(J) = C_n(N_{2,j+1}D_j - N_{2,j}D_{j+1}).
\end{align*}
\]

For all parameters of the queueing model given, it yields

\[
\begin{align*}
F(J) - F(J + 1) &= \begin{cases} 
  > 0, & \text{if } X(J) > Y(J), \\
  = 0, & \text{if } X(J) = Y(J), \\
  < 0, & \text{if } X(J) < Y(J).
\end{cases} \tag{77}
\end{align*}
\]

Thus, the monotonicity of \( F \) can be stated as

\[
\begin{align*}
F(J) \text{ is} \begin{cases} 
  \text{decreasing in } J, & \text{if } X(J) > Y(J), \\
  \text{constant in } J, & \text{if } X(J) = Y(J), \\
  \text{increasing in } J, & \text{if } X(J) < Y(J).
\end{cases} \tag{78}
\end{align*}
\]

Summarizing the above results, Corollary 1 can be obtained.

**Corollary 1.** Let \( J^* \) denote the optimal value of \( J \) which minimizes the expected cost \( F(J) \). We have

(i) If \( X(J) > Y(J) \), then \( J^* = \infty \);

(ii) If \( X(J) = Y(J) \), then \( J^* \) is any positive integer; and

(iii) If \( X(J) < Y(J) \), then \( J^* = 0 \).

**Remark 1:** It is interesting to mention that in practical applications, \( J^* = \infty \) can be referred to as a largest possible value of \( J \).
8. Numerical Illustrations

This section first presents some numerical results to study the effect of the parameter \( J \) on \( F(J) \). We consider three different vacation-time distributions: (i) exponential distribution (denoted by \( M \)) with LST \( \tilde{V}(\theta) = \frac{\lambda}{\theta + \gamma} \); (ii) 3-stage Erlang distribution (denoted by \( E_3 \)) with LST \( \tilde{V}(\theta) = (\frac{3\theta}{\theta + \gamma})^3 \); and (iii) deterministic distribution (denoted by \( D \)) with LST \( \tilde{V}(\theta) = e^{-\lambda \theta} \). Moreover, our numerical computations are based on the following assumptions:

- \( C_w = 10 \);  
- \( C_s = 600 \);  
- \( \lambda = 0.4 \);  
- \( \alpha = 0.05 \);  
- The batch size follows a geometric distribution \( Geo(1/3) \) with the first two moments \( E[X] = 3 \) and \( E[X^2] = 15 \);  
- All \( p_i \) (\( i = 0,1,\ldots,J \)) are the same;  
- All \( V_i \) (\( i = 0,1,\ldots,J \)) have identical distributions;  
- The service time obeys a 4-stage Erlang distribution with LST \( \tilde{V}(\theta) = (\frac{4\theta}{\theta + \gamma})^4 \);  
- The retrial time distribution is exponential with rate \( \gamma = 2.0 \) and LST \( R(\theta) = \frac{\gamma}{\theta + 2\gamma} \);  
- The repair time is 2-stage hyper-exponentially distributed with LST \( D(\theta) = q_1 \cdot \frac{\beta_1}{\theta + \beta_1} + q_2 \cdot \frac{\beta_2}{\theta + \beta_2} \), where we choose \( q_1 = 1/4 \), \( q_2 = 3/4 \), \( \beta_1 = 6.0 \) and \( \beta_2 = 9.0 \).  

All the parameter values are selected to satisfy the steady-state condition \( \rho_0 + E[X](1-R(\lambda)) < 1 \). Figures 1-2 depict the expected cost function \( F(J) \) with respect to \( J \) for the cases of \( X(J) > Y(J) \) and \( X(J) < Y(J) \), respectively. From Figure 1, we observe that (i) \( F(J) \) increases in \( J \) for three vacation-time distributions; (ii) the minimum of \( F(J) \) is achieved when \( J \) is sufficiently large; and (iii) \( F(J) \) for the exponential vacation-time distribution is smaller than that for other vacation-time distributions. Figure 2 shows that (i) \( F(J) \) increases with increasing value of \( J \) for three vacation-time distributions; (ii) the optimal value of \( J \) is zero, i.e., \( J^* = 0 \), for three vacation-time distributions; and (iii) \( F(J) \) for the exponential vacation-time distribution is larger than that for other vacation-time distributions. Moreover, it is noted that the numerical results are consistent with Corollary 1 in Section 7.

Next, we study the effects of \( J \) on \( L_q \) and \( E[T_r] \) for three vacation-time distributions. Numerical results are shown in Figures 3-4 with \( v = 5.0 \). As shown in Figure 3, one can find that (i) \( L_q \) increases as \( J \) increases; and (ii) the comparison of \( L_q \) for
three curves corresponding to three vacation-time distributions shows $L_q(M) > L_q(E_3) > L_q(D)$. From Figure 4, it reveals that (i) $E[T_i]$ increases with increasing values of $J$; and (ii) the comparison of $E[T_i]$ for three vacation-time distributions shows $E[T_i](M) > E[T_i](E_3) > E[T_i](D)$.

Figure 1. The effect of different values of $J$ on $F(J)$ for three vacation-time distributions in the case of $X(J) > Y(J)$.

Figure 2. The effect of different values of $J$ on $F(J)$ for three vacation-time distributions in the case of $X(J) < Y(J)$.
Figure 3. The effect of different values of $J$ on $L_q$ for three vacation-time distributions.

Figure 4. The effect of different values of $J$ on $E[T_r]$ for three vacation-time distributions.

9. Conclusions

In this paper, an $M^{[x]}/G/1$ retrial queue with general retrial times and $J$ optional vacations policy was proposed, where the server is subject to breakdowns and repairs. For this retrial queue, we derived the explicit expressions for the PGFs of the system size at a random epoch and at a departure epoch. Furthermore, we showed the existence of the stochastic decomposition property of the system size distribution for this queueing model. Some system performance measures and two reliability indices were also obtained. A cost model was constructed to determine the optimal value of $J$ to minimize the expected cost per unit time. Finally, we performed numerical examples to illustrate the influences of the
parameter $J$ on the expected cost function, expected number of customers in the orbit, and expected length of the busy cycle for three vacation-time distributions. In the future, we could extend this unreliable retrial queue to other types of repairs when a breakdown occurs, such as multi-optional repair or delaying repair.

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References


