Using Algebraic Approach to Solve an EPQ Model with Rework and Service Level Constraint

Maw-Liann Shyu¹, Yung-Fu Huang²,* and Chih-Sung Lai³

¹,³Department of Business Administration, Chaoyang University of Technology, Taichung 41349, Taiwan, R.O.C.
²Department of Marketing and Logistics Management, Chaoyang University of Technology, Taichung 41349, Taiwan, R.O.C.
*Corresponding author:E-mail: huf@cyut.edu.tw

Abstract- Chiu et al. [Mathematical and Computational Applications, Vol. 11, No. 1, pp. 75-84] examined the economic production quantity (EPQ) model with rework and service level constant. In this note, we will offer a simple algebraic approach to replace their differential calculus skill to find the optimal solution under the expected annual total cost minimized.

Keywords- Manufacturing, Economic production quantity, Service level constraint, Reworking

1. INTRODUCTION

One of the important manufacturing strategy to most manufacturing firms, is to be the low cost producer. That is, a firm must be able to effectively utilize its resources and minimize overall production costs. The Economic Production Quantity (EPQ) model is often employed practically in determining production lot size that minimizes total inventory costs. The EPQ can be considered an extension to the well-known Economic Order Quantity (EOQ) model, when items are produced internally instead of are
obtained from an outside supplier. It is also referred as a finite production model with assumption that the production rate is much larger than the demand rate. The EOQ (Economic Order Quantity) model is widely used by practitioners as a decision-making tool for the control of inventory. However, the assumptions of the EOQ model are rarely met. This has led many researchers to study the EOQ extensively under realistic situations. For minimizing the total relevant costs, in most previous all published papers that have been derived using differential calculus to find the optimal solution and the need to prove optimality condition with second-order derivatives. Grubbström and Erdem [4] and Cárdenas-Barrón [1] showed that the formulae for the EOQ and EPQ with backlogging derived without differential calculus. This algebraic approach could therefore be used easily to introduce the basic inventory theories to younger students. But Ronald et al. [5] thought that their algebraic procedure is too sophisticated to be absorbed by ordinary readers. Hence, Ronald et al. [5] derived a procedure to transform a two-variable problem into two steps, and then, in each step, they solve a one-variable problem using only the algebraic method without referring to calculus. Recently, Chang et al. [2] rewrote the objective function of Ronald et al. [5] such that the usual skill of completing the square can handle the problem without using their sophisticated method.

Recently, we study the paper of Chiu et al. [3] that investigated the EPQ model with rework and service level constant using differential calculus to find the optimal solution and the need to prove optimality condition with second-order derivatives. Therefore, in this note, we will offer a simple algebraic approach same as Chang et al. [2] to replace their differential calculus skill. This method can be offered another approach to study the basic inventory theories to readers.

2. ALGEBRAIC APPROACH IN CHIU ET AL’S MODEL [3]

For convenience, we adopt the same notation and assumptions as Chiu et al. [3] in this note.

Notation:

\( \lambda \) = demand rate in units per unit time,

\( P \) = production rate in units per unit time,
\( P_I = \) rate of rework of defective items in units per unit time,
\( x = \) the percentage of imperfect quality items produced; a random variable with a known probability density function,
\( d = \) production rate of the defective items in units per unit time,
\( Q = \) production lot size in the EPQ model with shortage not permitted,
\( Q_b = \) production lot size per cycle in the EPQ model with shortage allowed and backordered,
\( B = \) allowable backorder level in the EPQ model with backlogging permitted,
\( K = \) fixed setup cost for each production run,
\( c = \) production cost per item ($/item, inspection cost per item is included),
\( h = \) holding cost per item per unit time ($/item/unit time),
\( h_1 = \) holding cost for each imperfect quality item being reworked per unit time ($/item/unit time),
\( b = \) backordering cost per item per unit time,
\( c_R = \) repair cost for each imperfect quality item reworked ($/item),
\( T = \) the cycle length,
\( E[TCU(Q)] = \) the expected total inventory costs per unit time in the EPQ model with shortage not permitted,
\( E[TCU(Q_b, B)] = \) the expected total inventory costs per unit time in the EPQ model with backlogging permitted,

### 2.1. The EPQ Model with Rework and Shortage Not Permitted

From Eq. (13) in Chiu et al. [3], we know the expected annual cost, \( E[TCU(Q)] \), can be expressed as

\[
E[TCU(Q)] = \lambda [c + c_R E[x]] + \frac{K\lambda}{Q} + \frac{hQ}{2} \left(1 - \frac{\lambda}{P}\right) + \frac{\lambda Q}{2P_1} (h_1 - h) E[x^2] \\
= A + \frac{C}{Q} + DQ
\]  

(\( 1 \))

where the constants \( A = \lambda [c + c_R E[x]] \), \( C = K\lambda \) and \( D = \left[ \frac{h}{2} \left(1 - \frac{\lambda}{P}\right) + \frac{\lambda h_1}{2P_1} (h_1 - h) E[x^2] \right] \).
Our goal is to find the minimum solution of $E[TCU(Q)]$ by algebraic approach. Then we rewrite Eq. (1) as

$$E[TCU(Q)] = A + \frac{C}{Q} \left[ \sqrt{\frac{D}{C}} Q - 1 \right]^2 + 2\sqrt{CD}. \quad (2)$$

It implies that when

$$Q^* = \sqrt[2]{\frac{C}{D}} = \sqrt{\frac{2K\lambda}{h\left(1 - \frac{\lambda}{P}\right) + \frac{\lambda(h_h - h)}{P_h}E[x^2]}}. \quad (3)$$

Eq. (3), in this note, is the same as Eq. (14) in Chiu et al. [3].

### 2.2. The EPQ Model with Rework and Backlogging

From Eq. (9) in Chiu et al. [3], we know the expected annual cost, $E[TCU(Q_b, B)]$, can be expressed as

$$E[TCU(Q_b, B)] = \lambda[c + c_xE[x]] + \frac{K\lambda}{Q_b} + \frac{h}{2} \left( 1 - \frac{\lambda}{P} \right) Q_b - 2B$$

$$+ \frac{\lambda Q_h}{2P_l} (h_h - h) E[x^2] + \frac{B^2}{2Q_b} (b + h) E \left[ \frac{1 - x}{1 - x - \frac{\lambda}{P}} \right]$$

$$= A + \frac{C}{Q_b} + DQ_b + \frac{FB^2}{Q_b} + GB$$

where the constants $A$, $C$ and $D$ are the same as before case, and the constants $F$ and $G$ are given by

$$F = \frac{(b + h)}{2} E \left[ \frac{1 - x}{1 - x - \frac{\lambda}{P}} \right] \quad \text{and} \quad G = -h.$$

Our goal is to find the minimum solution of $E[TCU(Q_b, B)]$ by algebraic approach. Then we rewrite Eq. (4) as

$$E[TCU(Q_b, B)] = A + \frac{C}{Q_b} + \frac{F}{Q_b} \left[ B + \frac{GQ_h}{2F} \right]^2 + \left[ D - \frac{G^2}{4F} \right] Q_b. \quad (5)$$

It implies that when $Q_b$ is given, we can set $B$ as
\[ B = -\frac{G}{2F}Q_b \] to get the minimum value of \( E[TCU(Q_b, B)] \) as follows:

\[
E[TCU(Q_b, B(Q_b))] = A + \frac{C}{Q_b} + \left[ D - \frac{G^2}{4F} \right]Q_b. 
\] (6)

Then we rewrite Eq. (6) as

\[
E[TCU(Q_b, B(Q_b))] = A + \left[ \sqrt{\left(D - \frac{G^2}{4F}\right)Q_b} - \frac{C}{\sqrt{Q_b}} \right]^2 + 4C\left(D - \frac{G^2}{4F}\right). 
\] (7)

Then we can obtain the optimal production lot size

\[
Q_b^* = \sqrt{\frac{C}{D - \frac{G^2}{4F}}} = \sqrt{\frac{2K\lambda}{h\left(1 - \frac{\lambda}{P}\right) + \frac{\lambda(h - h)}{P}E[x^2] - \frac{h^2}{(b + h)E\left(\frac{1 - x}{1 - x - \frac{\lambda}{P}}\right)}}. 
\] (8)

and the optimal allowable backorder level

\[
B^* = -\frac{G}{2F}Q_b^* = \left(\frac{h}{b + h}\right)\frac{1}{E\left(\frac{1 - x}{1 - x - \frac{\lambda}{P}}\right)}Q_b^*. 
\] (9)

Eqs. (8) and (9), in this note, are the same as Eqs. (11) and (12) in Chiu et al. [3], respectively. Our procedure avoids the differential calculus to find the optimal solution and the need to prove optimality condition with second-order derivatives. We think this method can be easily accepted for readers and may be used to introduce the basic inventory theories to younger students.

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