Bounds on Optimal Cycle Time for The EOQ Model with Defective Items Taking Account of Time Value

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Abstract

This paper follows the discounted cash flow (DCF) approach to investigate inventory replenishment problem for EOQ model with defective items taking account of time value of money. First, this paper shows that the present value of the total variable cost for the infinite planning horizon is a convex function. Second, this paper derives bounds to determine the optimal cycle time under minimum value of the present value of the total variable cost for the infinite planning horizon. Third, with these bounds, a simple algorithm is derived to help the determination of the exact optimal cycle time and optimal order quantity. Finally, a numerical example is given to illustrate the algorithm discussed in this paper.

Keywords: EOQ, DCF, defective item, present value, time value of money
INTRODUCTION

The EOQ model is widely used by practitioners as a decision-making tool for the control of inventory. However, the assumptions of the EOQ model are rarely met. This has led many researchers to study the EOQ extensively under realistic situations.

A common unrealistic assumption of the EOQ is that all units produced or purchased are of good quality. We know that it is difficult to produce or purchase items with 100% good quality. Recently, Salameh and Jaber\(^1\) extended the classical EOQ model by accounting for imperfect quality items when using the EOQ formulate. They also considered the issue that poor-quality items are sold as a single batch by the end of the 100% screening process. Schwaller\(^2\) presented a procedure to extend EOQ models by adding the assumptions that a known proportion of defectives that must be removed are present is examined. Cheng\(^3\) proposed an EOQ model with demand-dependent unit production cost and imperfect production process. Zhang and Gerchak\(^4\) considered a joint lot sizing and inspection policy with random yield where the defective units cannot be used and thus must be replaced by non-defective ones.

In all of the above literatures mentioned, the time value of money was ignored. So, another unrealistic assumption of classical EOQ model is that all the costs associated with the inventory system remain constant over time. Perhaps this is because of difficulty of discounting the periodic cash flows that occur in inventory systems. Often, it has been claimed that the time value of money would not influence the inventory policy variables too significant, particularly if the interest rates are low and the planning horizon is short. However, if the interest rates currently are not low and the planning horizon may not short. Inventory policies obviously affect profitability and any particular policy choice depends upon relative profitability. A measure of profitability that disregards the time value of money has the advantage of simplicity, but can yield very misleading results. So, it has not been possible to ignore the effects of the time value of money any further. Therefore, it is preferable to consider improving validity of the analysis. There are many researchers
to reformulate the optimal inventory management policies taking into account time value of money, e.g. Buzacott,5 Bierman and Thomas,6 Misra and Wortham,7 Misra,8,9 Aggarwal,10 Gurnani,11 Chandra and Bahner,12 Datta and Pal,13 Dohi et al.,14 Bose et al.,15 Chung,16 Chung et al.,17 Chung et al.,18 Chung,19 Chung and Huang20 and their references.

The main purpose of this paper is to want to modify above two unrealistic assumptions of EOQ model to develop the inventory system to allow defective items and take account of time value of money. This paper will adopt 100% inspection policy when the items are received. Besides, the known proportion of defective items is sold prior to storage or use after the screening process. Moreover, the purpose of this paper is threefold: first, this paper shows that the present value of the total variable cost for the infinite planning horizon is a convex function; second, this paper derives bounds to determine the optimal cycle time under minimum value of the present value of the total variable cost for the infinite planning horizon; third, with these bounds, a simple algorithm is derived to help the determination of the exact optimal cycle time and optimal order quantity. Finally, a numerical example is given to illustrate the algorithm discussed in this paper.

**MODEL FORMULATION**

The following notations and assumptions will be used throughout:

**Notations**

\[ Q = \text{order quantity} \]
\[ D = \text{annual demand} \]
\[ A = \text{cost of placing one order} \]
\[ i = \text{unit inspection cost} \]
\[ I = \text{the fixed inspection cost incurred with each lot} \]
\[ h = \text{unit stock holding cost per item per year} \]
\[ k = \text{the known percentage of defective items in } Q \]

\[ r = \text{the discount rate per unit time} \]

\[ T = \text{the cycle time} \]

\[ TVC(T) = \text{the present value of the total variable cost for the first inventory cycle} \]

\[ TVC_{\infty}(T) = \text{the present value of the total variable cost for the infinite planning horizon} \]

\[ T^* = \text{the optimal cycle time of } TVC_{\infty}(T) \]

\[ Q^* = \text{the optimal order quantity} = \frac{DT^*}{1-k}. \]

**Assumptions**

1. Demand rate is known and constant.
2. Shortages are not allowed.
3. Time period is infinite.
4. The ordering lead time is zero.
5. A lot of size \( Q \) is delivered instantaneously.
6. Each lot received contains a known proportion of defectives that sold prior to storage or use.
7. The screening time for items can be neglect. The inspection cost consists of a fixed per lot inspection cost and a fixed unit inspection cost.
8. We use continuous discounting for expositional convenience; the discount factor will become \( e^{-rt} \) at time \( t \).

The total variable cost for the first inventory cycle consists of the following elements:

1. At the beginning of each inventory cycle, there will be ordering cost = \( A \), fixed per lot inspection cost = \( I \) and unit inspection cost = \( \frac{iDT}{1-k} \).
(2) The stock holding cost per unit time at \( t = hD(T-t) \). Then the present value of the stock holding cost = \( hD(T-t) e^{-rt} \). Thus the present value of the stock holding cost for the first inventory cycle =

\[
hD \int_0^T (T-t)e^{-rt} dt.
\]

So, the present value of the total variable cost for the first inventory cycle is

\[
TVC(T) = A + I + \frac{idT}{1-k} + hD \int_0^T (T-t)e^{-rt} dt. (1)
\]

Note that

\[
\int_0^T (T-t)e^{-rt} dt = \frac{1}{r} \left[ T + \frac{1}{r} (e^{-rT} - 1) \right]. (2)
\]

Substituting equation (2) into equation (1), the present value of the total variable cost for the first inventory cycle is

\[
TVC(T) = A + I + \frac{idT}{1-k} + hD \left[ T + \frac{1}{r} (e^{-rT} - 1) \right]. (3)
\]

Then the present value of the total variable cost for the infinite planning horizon \( TVC_\infty(T) \) is

\[
TVC_\infty(T) = TVC(T) \sum_{n=0}^{\infty} e^{-nrT}. (4)
\]

Using

\[
\sum_{n=0}^{\infty} e^{-nrT} = \frac{1}{1-e^{-rT}}, (5)
\]

and substituting equation (3) into equation (4), we obtain

\[
TVC_\infty(T) = -\frac{hD}{r^2} + \frac{A + I}{1-e^{-rT}} + \frac{\frac{ir}{1-k} + h)DT}{r(1-e^{-rT})}. (6)
\]
CONVEXITY OF $TVC_\alpha(T)$

In this section, we want to show the present value of the total variable cost for the infinite planning horizon $TVC_\alpha(T)$ is a convex function. Before proving that, we need following lemma 1.

**Lemma 1:**

(A) \( \frac{1}{1-e^{-rt}} \) is convex for \( T > 0 \).

(B) \( \frac{T}{1-e^{-rt}} \) is convex for \( T > 0 \).

**Proof:**

(A) We can obtain that

\[
\frac{d}{dT}\left( \frac{1}{1-e^{-rt}} \right) = -\frac{re^{-rt}}{(1-e^{-rt})^2}
\]

and

\[
\frac{d^2}{dT^2}\left( \frac{1}{1-e^{-rt}} \right) = \frac{r^2e^{-rt}}{(1-e^{-rt})^2} + \frac{2r^2e^{-2rt}}{(1-e^{-rt})^3} \geq 0 .
\]

Because \( rT > 0 \) and \( \frac{d^2}{dT^2}\left( \frac{1}{1-e^{-rt}} \right) \geq 0 \) for all \( T > 0 \). Hence, \( \frac{1}{1-e^{-rt}} \) is convex for \( T > 0 \).

(B) We can obtain that

\[
\frac{d}{dT}\left( \frac{T}{1-e^{-rt}} \right) = \frac{1-e^{-rt}-rTe^{-rt}}{(1-e^{-rt})^2}
\]

and

\[
\frac{d^2}{dT^2}\left( \frac{T}{1-e^{-rt}} \right) = \frac{re^{-rt}(rT + rTe^{-rt} + 2e^{-rt} - 2)}{(1-e^{-rt})^3} .
\]

Let \( h(T) = (rT + rTe^{-rt} + 2e^{-rt} - 2) \) for \( T > 0 \). Then \( h'(T) = r(1-e^{-rt} - rTe^{-rt}) , \)

\( h(0) = 0 \) and \( h''(T) = r^3Te^{-rt} \). Because \( h''(T) > 0 \) for \( T > 0 \), \( h'(T) \) is increasing on \( T \).
> 0 and \( h'(T) > h'(0) = 0 \). So \( h(T) \) is increasing on \( T > 0 \). Hence, \( h(T) > h(0) = 0 \). So

\[
(rT + rTe^{-rT} + 2e^{-rT} - 2) > 0 \quad \text{for} \quad T > 0.
\]

Therefore, \( \frac{d^2}{dT^2} \left( \frac{T}{1 - e^{-rT}} \right) > 0 \) for \( T > 0 \).

Consequently, \( \frac{T}{1 - e^{-rT}} \) is convex for \( T > 0 \).

From lemma 1, we have the following theorem.

**Theorem 1**: The present value of the total variable cost for the infinite planning horizon \( TVC_\infty(T) \) is convex.

**Proof**: From equation (6), we know

\[
TVC_\infty(T) = -\frac{hD}{r^2} + \frac{A + I}{1 - e^{-rT}} + \frac{(ir + h)DT}{r(1 - e^{-rT})} \quad \text{for} \quad T > 0.
\]

Rearrangement,

\[
TVC_\infty(T) = -\frac{hD}{r^2} + (A + I) \frac{1}{1 - e^{-rT}} + \left( \frac{ir + h}{1 - k} \right) \frac{T}{r(1 - e^{-rT})}.
\] (11)

From lemma 1, equation (11) implies that \( TVC_\infty(T) \) is convex for \( T > 0 \).

**BOUNDS FOR THE OPTIMAL CYCLE TIME**

In previous section, we have showed that the present value of the total variable cost for the infinite planning horizon \( TVC_\infty(T) \) is a convex function. By the convexity, now we will find the optimal cycle time under minimum value of \( TVC_\infty(T) \). This paper does not adopt approximation approach liking mostly literatures. We try to find the upper bound and lower bound of optimal cycle time \( T^* \), then present an algorithm to help us to easily and quickly decide the exact optimal cycle time.
Setting the derivative of equation (6) with respect to \( T \) yields

\[
TVC'_\infty (T) = \frac{\left( \frac{ir}{1-k} + h \right)D - \left( (A + I)r^2 + \left( \frac{ir}{1-k} + h \right)D(1+rT) \right)e^{-rT}}{r(1 - e^{-rT})^2}.
\]  

(12)

Let equation (12) equal to zero, we can get

\[
\left( \frac{ir}{1-k} + h \right)D - \left( (A + I)r^2 + \left( \frac{ir}{1-k} + h \right)D(1+rT) \right)e^{-rT} = 0.
\]

(13)

Equation (13) multiplied by \( e^{rT} \) yields

\[
D\left( \frac{ir}{1-k} + h \right)e^{rT} - \left( (A + I)r^2 + \left( \frac{ir}{1-k} + h \right)D(1+rT) \right) = 0.
\]

(14)

We let

\[
f(T) = D\left( \frac{ir}{1-k} + h \right)e^{rT} - \left( (A + I)r^2 + \left( \frac{ir}{1-k} + h \right)D(1+rT) \right).
\]

(15)

The optimal cycle time \( T^* \) satisfies

\[
f(T^*) = 0.
\]

(16)

Then \( f(T) \) is strictly increasing for \( T > 0 \). Therefore the optimal cycle time \( T^* \) is the unique positive solution of the equation (14). Let

\[
T^L = \frac{r(T^U)^2}{\left[ 2(e^{rT^U} - 1 - rT^U) \right]^2}
\]

(17)

and

\[
T^U = \frac{2(A + I)}{\sqrt{D(h + \frac{ir}{1-k})}}.
\]

(18)

Then we have the following results.

**Theorem 2:** \( T^L < T^* < T^U \).

**Proof:**

(i) From equation (15), we have
\[ f(T^U) = D \left( \frac{ir}{1-k} + h \right) e^{rT^U} - \left[ (A + I)r^2 + \left( \frac{ir}{1-k} + h \right) D(1 + rT^U) \right]. \] (19)

Since \( e^{rT^U} > 1 + rT^U + \frac{r^2(T^U)^2}{2} \) and \( T^U = \sqrt{\frac{2(A + I)}{D(h + \frac{ir}{1-k})}} \), so

\[
\begin{align*}
f(T^U) &> D \left( \frac{ir}{1-k} + h \right) \left( 1 + rT^U + \frac{r^2(T^U)^2}{2} \right) - \left[ (A + I)r^2 + \left( \frac{ir}{1-k} + h \right) D(1 + rT^U) \right] \\
&= (A + I)r^2 - (A + I)r^2 \\
&= 0 = f(T^*).
\end{align*}
\]

Since \( f(T) \) is strictly increasing, \( T^U > T^* \).

(ii) On the other hand, equation (16) is the optimal condition of the optimal cycle time \( T^* \), that is

\[
D \left( \frac{ir}{1-k} + h \right) e^{rT^*} - \left[ (A + I)r^2 + \left( \frac{ir}{1-k} + h \right) D(1 + rT^*) \right] = 0 \] (20)

Rearrangement,

\[
\frac{(A + I)r^2}{D \left( \frac{ir}{1-k} + h \right)} = e^{rT^*} - 1 - rT^* \\
= \frac{r^2T^*^2}{2} + \frac{r^3T^*^3}{6} + \frac{r^4T^*^4}{24} + \cdots
\]

Hence,

\[
\frac{2(A + I)}{D \left( \frac{ir}{1-k} + h \right)} = T^*^2 + \frac{rT^*^3}{3} + \frac{r^2T^*^4}{12} + \cdots. \] (21)

Insert equation (18) into equation (21),

\[
(T^U)^2 = T^*^2 + \frac{rT^*^3}{3} + \frac{r^2T^*^4}{12} + \cdots \] (22)

Equation (22) multiplied by \( 1/T^*^2 \) on both sides yields

\[
\frac{(T^U)^2}{T^*^2} = 1 + \frac{rT^*}{3} + \frac{r^2T^*^2}{12} + \cdots. \] (23)
Hence,

\[
\frac{T^U}{T^*} = \left(1 + \frac{r T^*}{3} + \frac{r^2 T^*^2}{12} + \cdots \right)^{\frac{1}{2}} \\
< \left(1 + \frac{r T^U}{3} + \frac{r^2 (T^U)^2}{12} + \cdots \right)^{\frac{1}{2}} = \frac{[2(e^{r T^U} - 1 - r T^U)]^\frac{1}{2}}{r T^U}.
\]

From equation (24), we can obtain

\[
T^* > \frac{r (T^U)^2}{[2(e^{r T^U} - 1 - r T^U)]^\frac{1}{2}} = T^L.
\]

Combining (i) and (ii), this completes the proof of Theorem 2. Consequently,

\[T^L < T^* < T^U.\]

THE ALGORITHM

Now, we will present an algorithm to get the optimal cycle time \(T^*\) for the present value of the total variable cost for the infinite planning horizon \(TVC_\infty(T)\). Before describing the algorithm, we need the following theorem.

**INTERMEDIATE VALUE THEOREM** [Thomas and Finney (1996)]^21: Let \(g\) be a continuous function on \([a, b]\) and let \(g(a)g(b) < 0\). Then there exists a number \(d\in(a, b)\) such that \(g(d)=0\).

Since \(f(T)\) is strictly increasing, the following algorithm is based on Theorem 2, Intermediate Value Theorem and the uniqueness of the root of equation (14). Now we are in a position to outline the algorithm to determine the optimal cycle time \(T^*\). Note that, \(f(T^U) > 0\) and \(f(T^L) < 0\).

**The algorithm**

Step 1: Let \(\varepsilon > 0\).
Step 2: Set $T_U = T^U$ and $T_L = T^L$.

Step 3: Set $T_{opt} = \frac{T_L + T_U}{2}$.

Step 4: If $|f(T_{opt})| < \varepsilon$, go to Step 6. Otherwise, go to Step 5.

Step 5: If $f(T_{opt}) > 0$, set $T_U = T_{opt}$. If $f(T_{opt}) < 0$, set $T_L = T_{opt}$. Then go to Step 3.

Step 6: Set $T^* = T_{opt}$, $Q^* = \frac{DT^*}{1-k}$ and exit the optimal cycle time and order quantity.

The above algorithm is really very simple and commonly known as bisection. We use the following numerical example to illustrate the algorithm.

**Example**: Let $A=$ $100/order, $D=1000units/year, $I=$ $10/order, $i=$ $0.1/unit, $h=$ $5/unit/year, $r=$ $0.05$/$/year, $k=2%/$order. Taking $\varepsilon = 0.0001$ and following the above algorithm, we can speedily and easily obtain the optimal cycle time $T^* = 0.20931143$ year and the optimal order quantity $Q^* = \frac{DT^*}{1-k} = 214$ units after four iterations.

**CONCLUSIONS**

This paper adopts the discounted cash flows (DCF) approach to explore the EOQ inventory model with defective items taking account of time value of money. First, we construct the present value of the total variable cost for the infinite planning horizon $TVC_c(T)$. Second, we show that $TVC_c(T)$ is a convex function. Third, this paper derives lower bound and upper bound to determine the optimal cycle time. Fourth, with these bounds, a simple algorithm to help the exact optimal cycle time is decided.

To simplify the process of the solution procedure, many researchers assume that $rT$ is sufficiently
small about equation (14). Then they use approximation, let $e^{rT} = 1 + rT + \frac{r^2T^2}{2}$, to deal with the complex solution procedure. But $rT$ is not necessarily small. At this condition, the approximation approach may cause significant errors and penalties. However accurate but complex methodology may defy practical implementation. So, there exists the motivation to present a simple, practical and accurate solution procedure to determine the optimal cycle time $T^*$. The bounds for $T^*$ provided by this paper will be useful and simple in search procedures for determining $T^*$. In practice, if production managers use the solution algorithm in this paper, they will find it simple, easy, rapid and accurate. From the point of implementation, the solution algorithm in this paper is rather beneficial.

Finally, a numerical example is given to illustrate the algorithm discussed in this paper. A future study should further incorporate the proposed model into more realistic assumptions, such as probabilistic demand, allowable shortages and a finite rate of replenishment.
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